ANALISI FUNZIONALE — A. Visintin (2013)

Questo corso introduce gli elementi essenziali dell'Analisi Funzionale, rinviando le applicazioni alle PDEs al relativo corso della laurea magistrale.

Prerequisiti

Calcolo differenziale ed integrale, con serie di Fourier ed ODE *(Analisi I, II e III).*
Teoria della misura di Lebesgue e dell'integrazione *(Analisi III).*
Algebra lineare *(Geometria I).*
Topologia generale *(Geometria II).*

PROGRAMMA DEL CORSO

1. Spazi di Banach

Spazi normati e di spazi di Banach. Esempi in dimensione finita. Spazi C^k.
Disuguaglianze di Young, Hölder e Minkowski. Spazi L^p e spazi di successioni. Teoremi di rappresentazione degli spazi L^p ed ℓ^p.
Teorema di Hahn-Banach, corollari e teoremi di separazione. Lemma di Riesz e caratterizzazione degli spazi di dimensione finita.
Basi algebriche e topologiche. Spazio degli operatori lineari e continui tra spazi normati. Operatori limitati e continui.
Seminorme e spazi di Fréchet.
Spazio duale e biduale. Convergenze debole e debole star. Teorema di compattezza di Banach.

2. Spazi di funzioni continue

Funzioni continue. Convergenza uniforme. Teorema di convergenza di Dini.
Algebre di Banach e teorema di Stone-Weierstrass (enunciato).
Teorema di compattezza di Ascoli-Arzelà.
Spazi di Hölder.

3. Operatori I

Teorema di Baire.
Teorema di Banach-Steinhaus.
Teorema di Banach dell'applicazione aperta (enunciato).
Teorema del grafico chiuso.
Operatori aggiunti.
Operatori compatti ed esempi.

4. Spazi di Hilbert

Prodotto scalare. Proiezione ortogonale su un convesso chiuso.
Teorema di rappresentazione di Riesz-Fréchet. Teorema di Lax-Milgram.
Ortogonalizzazione di Gram-Schmidt. Insiemi ortonormali. Basi Hilbertiane. Serie di Fourier in spazi di Hilbert.

5. Operatori II

Teorema di Riesz (enunciato). Teorema dell'alternativa di Fredholm.
Spettro di un operatore. Spettro di un operatore compatto.

Esercitazioni

Queste tratteranno quesiti riguardanti la teoria sopra indicata. Particolare attenzione verrà dedicata alla discussione di esempi e controesempi. Numerosi esercizi sono disponibili su [Br] e [AV].

Testi di riferimento

[Br] A. Bressan: *Lecture Notes on Functional Analysis with Applications to Linear Partial Differential Equations.* American Mathematical Society, 2012 (Chaps. 2–5)

[AV] Note integrative del docente disponibili in rete.

Alcuni testi di consultazione

R. Bhatia: *Notes on Functional Analysis. (Lez. 1–15.)* Hindustan Book Agency, New Delhi 2009 [un testo introduttivo]

H. Brezis: *Functional Analysis, Sobolev Spaces and Partial Differential Equations.* Springer, New York 2011 [un testo avanzato, con numerosi esercizi svolti]

G. Teschl: *Topics in Real and Functional Analysis. (Part 1)* Note disponibili in rete. [un testo introduttivo]

Modalità di esame

Prova scritta con esercizi e quesiti di teoria, seguita da una prova orale.

Guida per una preparazione essenziale

1. Spazi di Banach. [Br]: chap. 2 + [AV]: spazi L^p, dimensione e basi.

2. Spazi di funzioni continue. [Br]: chap. 3.

3. Operatori I. [Br]: chap. 4 + [AV]: teorema di Baire.

4. Spazi di Hilbert. [Br]: chap. 5 + [AV]: proiezione sui convessi.

(Oppure: *4. Spazi di Hilbert.* [AV]: Hilbert Spaces, Orthogonality and Projections, The Representation Theorem, Orthonormal Systems and Hilbert Bases.)

5. Operatori II. [AV]: chap. Operators.

Notes to the course of Functional Analysis (2013)

A. Visintin

These notes complement Chaps. 2–5 of Bressan's book, and may not substitute it for the preparation of the exam.

By [Ex] we mean that the justification of a statement is left as an exercise.

By [] we mean that the justification is omitted, and is more than just an exercise.

By [Br] we indicate Bressan's book.

The asterisk is used to label complements in the text (in particular this applies to some slightly technical arguments), as well as more demanding exercises.

Contents

1 L^p Spaces

1.1 Three fundamental inequalities for L^p spaces

Lemma 1.1 (*Young Inequality*) *For any $p, q > 1$ such that $1/p + 1/q = 1$,*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \qquad \forall a, b \geq 0. \tag{1.1}$$

Proof. Without loss of generality we may assume that $a, b > 0$. By the concavity of the logarithm function we have

$$\log\Big(\frac{a^p}{p} + \frac{b^q}{q}\Big) \geq \frac{1}{p}\log(a^p) + \frac{1}{q}\log(b^q) = \log a + \log b = \log(ab)\,.$$

As the exponential function is monotone, by passing to the exponentials we get (1.1). □

Let $(A, \mathcal{A}, \mu)$ be a measure space with μ a positive measure, and denote by $\mathcal{M}(A, \mathcal{A}, \mu)$, or just $\mathcal{M}(A)$, the linear space of equivalence classes of μ-a.e. coinciding measurable functions $A \to \mathbb{K}$. The sets

$$\begin{aligned} &L^p(A) := \{v \in \mathcal{M}(A) : \|v\|_p := \Big(\int_\Omega |v(x)|^p \, d\mu(x)\Big)^{1/p} < \infty\} \;\; (0 < p < \infty), \\ &L^\infty(A) := \Big\{v \in \mathcal{M}(A) : \|v\|_\infty := \operatorname*{ess\,sup}_\Omega |v| < \infty\Big\}, \end{aligned} \tag{1.2}$$

where $\operatorname{ess\,sup}_\Omega |v| := \inf_{\mu(N)=0} \sup_{x \in \Omega \setminus N} |v(x)|$, are linear subspaces of $\mathcal{M}(A)$.

Theorem 1.2 (*Hölder Inequality) For any $p, q \in [1, +\infty]$ with $1/p + 1/q = 1$,* [1]

$$uv \in L^1(A), \qquad \int_A |u(x)v(x)|\, d\mu(x) \le \|u\|_p \|v\|_q \qquad \forall u \in L^p(A), \forall v \in L^q(A). \tag{1.3}$$

Proof. We may assume that $u, v \not\equiv 0$ (the null function) and that both p and q are finite and different from 1, since otherwise the result is trivial. After replacing u by $\widetilde{u} := u/\|u\|_p$ and v by $\widetilde{v} := v/\|v\|_q$, we are reduced to proving that

$$\widetilde{u}\widetilde{v} \in L^1(A), \qquad \int_A |\widetilde{u}(x)\widetilde{v}(x)|\, d\mu(x) \le 1 \qquad \forall u \in L^p(A), \forall v \in L^q(A). \tag{1.4}$$

The Young inequality (1.1) yields

$$|\widetilde{u}(x)\widetilde{v}(x)| \le \frac{1}{p}|\widetilde{u}(x)|^p + \frac{1}{q}|\widetilde{v}(x)|^q \qquad \text{for a.e. } x \in A.$$

Integrating over A we get $\widetilde{u}\widetilde{v} \in L^1(A)$ and

$$\int_A |\widetilde{u}(x)\widetilde{v}(x)|\, d\mu(x) \le \frac{1}{p}\int_A |\widetilde{u}(x)|^p\, d\mu(x) + \frac{1}{q}\int_A |\widetilde{v}(x)|^q\, d\mu(x) = \frac{1}{p} + \frac{1}{q} = 1\,,$$

that is (1.4). □

Proposition 1.3 (*Minkowski Inequality) For any $p \in [1, +\infty]$,*

$$\|u + v\|_p \le \|u\|_p + \|v\|_p \qquad \forall u, v \in L^p(A). \tag{1.5}$$

Proof. We may confine ourselves to the case $1 < p < +\infty$, for otherwise the statement is obvious. For a.e. $x \in A$,

$$|u(x) + v(x)|^p \le 2^p(\max\{|u(x)|, |v(x)|\})^p \le 2^p\big(|u(x)|^p + |v(x)|^p\big)\,;$$

by integrating over A we may conclude that $u+v \in L^p(A)$. Setting $q := p/(p-1)$, by the Hölder inequality we have

$$\begin{aligned}
\|u+v\|_p^p &= \int_A |u(x)+v(x)||u(x)+v(x)|^{p-1}\, d\mu(x) \\
&\le \int_A |u(x)||u(x)+v(x)|^{p-1}\, d\mu(x) + \int_A |v(x)||u(x)+v(x)|^{p-1}\, d\mu(x) \\
&\le \big(\|u\|_p + \|v\|_p\big)\|\,|u+v|^{p-1}\|_q = \big(\|u\|_p + \|v\|_p\big)\|u+v\|_p^{p-1}.
\end{aligned}$$

This yields the desired inequality. □

The Minkowski inequality is the triangular inequality for L^p spaces, which are thus normed spaces. We shall see that these spaces are also complete, so that they are Banach spaces.

Discrete inequalities. Selecting $A = \{1, \dots, M\}$ and μ equal to the counting measure, the Hölder inequality provides a discrete version for finite sums:

$$\sum_{n=1}^{M} |a_n b_n| \le \left(\sum_{n=1}^{M} |a_n|^p\right)^{1/p} \left(\sum_{n=1}^{M} |b_n|^q\right)^{1/q} \tag{1.6}$$

[1] Here and in the following, this means that $q = \infty$ if $p = 1$, and $p = \infty$ if $q = 1$.

for any $a_1, ..., a_M, b_1, ..., b_M \in \mathbb{C}$ and any $M \in \mathbb{N}$. Passing to the limit, one then gets the Hölder inequality for ℓ^p spaces.

Similarly, a discrete version of the Minkowski inequality provides the triangular inequality for ℓ^p spaces.

1.2 Nesting of L^p and ℓ^p spaces

We claim that for any μ-measurable set A of finite measure and any $p, q \in [1, +\infty]$

$$1 \le p \le q < +\infty \qquad \Rightarrow \qquad L^q(A) \subset L^p(A). \tag{1.7}$$

This inclusion is easily checked if either p or q equals either 1 or $+\infty$. Let us then assume that $p, q \in \,]1, +\infty[$. Notice that the exponents $r = q/p$ and $s = q/(q-p)$ are conjugate. For any $v \in L^q(A)$, the Hölder inequality then yields

$$\|v\|_p^p = \int_A |v(x)|^p \cdot 1 \, d\mu(x) \le \| \, |v(x)|^p \, \|_r \, \|1\|_s = \|v(x)\|_q^p \, \mu(A)^{1/s}.$$

This proves the claim. This inequality actually shows more: the injection $L^q(A) \to L^p(A)$ is continuous. This means that if a sequence converges in $L^q(A)$, then it converges to the same limit in $L^p(A)$.

The analogous statement fails if $\mu(A) = +\infty$. For instance, let $A = \,]-1, +\infty[$, $1 \le p \le q < +\infty$ and $\alpha < 0$ be such that $\alpha q < -1 \le \alpha p$; then $x^\alpha \in L^q(A) \setminus L^p(A)$. Notice that there exists always such an α, whenever p, q are as above.

For ℓ^p spaces the inclusions are reversed:

$$1 \le p \le q \le +\infty \qquad \Rightarrow \qquad \ell^p \subset \ell^q. \qquad \text{[Ex]} \tag{1.8}$$

Why is there this reversal? This may be understood considering that functions defined on a finite measure set may have a large L^p-norm only if somewhere they are large. On the other hand sequences may have a large ℓ^p-norm only if they do not decay sufficiently fast. Moreover the behaviors of powers of large real numbers is opposite to that of small values: as p increases, x^p increases for any $x > 1$, and instead decreases for any $0 < x < 1$.

1.3 Properties of L^p and ℓ^p spaces

These spaces play an important role in functional analysis, since are a large source of examples and counterexamples. Here we state some of their properties without proofs.

From now on, our measure space $(A, \mathcal{A}, \mu)$ will be a (possibly unbounded) Euclidean open set (i.e., an open subset of $\mathbb{R}^N$ for some integer N), denoted by Ω, equipped with the standard Lebesgue measure on the Borel σ-algebra. The following is a classical result of measure theory, and is at the basis of the importance of these spaces for analysis. [2]

Theorem 1.4 (*Fischer-Riesz*) *For any $p \in [1, +\infty]$, the normed space $L^p(\Omega)$ is complete. []*

L^p-spaces are thus Banach spaces.

We shall denote by $C^0_c(\Omega)$ the linear space of compactly supported continuous functions $\Omega \to \mathbb{R}$; this is obviously a linear subspace of $L^p(\Omega)$ for any $p \in [1, +\infty]$.

Theorem 1.5 (*Density*) *For any $p \in [1, +\infty[$, the linear space $C^0_c(\Omega)$ is dense in $L^p(\Omega)$. This fails for $L^\infty(\Omega)$. []*

[2] For a short review of the Lebesgue measure theory see the Appendix of [Br]. A more extended treatment of L^p spaces may be found, e.g., in Chap. 4 of [Bz].

(This may be proved via the classical Lusin theorem.)

On the other hand, $C^0_c(\Omega)$ is not dense in $L^\infty(\Omega)$. For instance, the sign function ($S(x) = -1$ if $x < 0$, $S(x) = 1$ if $x > 0$) cannot be approximated by continuous functions in the metric of this space.

Theorem 1.6 *For any $p \in [1,+\infty[$, the space $L^p(\Omega)$ is separable (i.e., it has a countable dense subset).* [3] *This fails for $L^\infty(\Omega)$. []*

In $L^\infty(\Omega)$ counterexamples are easily constructed.

Theorem 1.7 (*Fréchet-Riesz) Let $p \in\ ]1,+\infty[$ and set $p' = p/(1-p)$ (this is the conjugate exponent). Then*

$$[\Phi_p(f)](v) := \int_\Omega fv\,dx \qquad \forall f \in L^{p'}(\Omega), \forall v \in L^p(\Omega) \tag{1.9}$$

defines an isometric isomorphism $\Phi_p : L^{p'}(\Omega) \to L^p(\Omega)'$. []

Next we consider the dual space of $L^1(\Omega)$ and of $L^\infty(\Omega)$.

Theorem 1.8 (*Steinhaus-Nikodým) $\Phi_1 : L^\infty(\Omega) \to L^1(\Omega)'$ is an isometric isomorphism. []*

Proposition 1.9 *Define the operator $\Phi_\infty : L^1(\Omega) \to L^\infty(\Omega)'$ as in (1.9), with $\infty' = 1$. This is a nonsurjective isometry. []*

Conclusions. For any $p \in [1,+\infty[$ ($p = \infty$ excluded), denoting the conjugate index by p', we may identify $L^p(\Omega)'$ with $L^{p'}(\Omega)$. This fails for $p = \infty$, since we may just identify $L^1(\Omega)$ with a proper closed subspace of $L^\infty(\Omega)'$.

More generally, the same holds for any measure space $(A, \mathcal{A}, \mu)$: [4] we may thus identify $(\ell^p)'$ with $\ell^{p'}$ for any $p \in [1,+\infty[$, but not for $p = \infty$.

ℓ^1 **has a predual.** With standard notation, we shall denote by c the set of converging sequences, by c_0 the set of sequences that tend to 0, and by c_{00} the set the sequences that have just a finite number of nonvanishing terms. All of these are normed subspaces of ℓ^∞.

Although in general $L^1(\Omega)$ need not have a predual, the following result allows one to identify ℓ^1 with $(c_0)'$. (Thus ℓ^1 is the dual of a separable Banach space.)

Proposition 1.10 *Let us set*

$$[\phi(u)](v) := \sum_{n=0}^{\infty} u_n v_n \qquad \forall u = \{u_n\} \in \ell^1, \forall v = \{v_n\} \in c_0. \tag{1.10}$$

This defines ϕ as a surjective isometry $\ell^1 \to (c_0)'$. []

1.4 Exercises

1. (i) For any $p \in [1,2]$, prove that $L^p(\mathbb{R}) \subset L^1(\mathbb{R}) + L^2(\mathbb{R})$.
   (ii) More generally, for any $p, q, r \in [1,+\infty]$, prove that if $p < q < r$ then $L^q(\mathbb{R}) \subset L^p(\mathbb{R}) + L^r(\mathbb{R})$.

[3] * This may be derived from the classical Weierstraß theorem, that states the linear space of polynomial is dense in the space of continuous functions. By modifying the polynomials, one easily constructs a countable family that approximates all compactly supported functions on Ω. By the above density theorem, and as the L^p-norm is dominated by the uniform norm, one easily concludes that this set of functions is dense in $L^p(\Omega)$.

[4] for $p = 1$ (and just for this index) the measure should be assumed σ-finite; that is, the set A should be representable as a countable union of sets of finite measure.

2 Banach Spaces

*2.1 Ordered sets

Order structures are sometimes used in analysis, especially via the Zorn lemma or also via the Hausdorff's maximal chain theorem. Here we briefly review those structures.

A set S equipped with a binary relation $\leq$ is called *(partially) ordered* iff

$$x \leq x, \quad x \leq y,\ y \leq x \Rightarrow x = y, \quad x \leq y,\ y \leq z \Rightarrow x \leq z \qquad \forall x, y, z \in S.$$

(If the second of these properties (i.e., the antisymmetry) is not fulfilled, $\leq$ is called a *pre-order.* In this case an order relation is obtained by taking the quotient w.r.t. [5] the equivalence relation defined by $x \equiv y$ iff $x \leq y$ and $y \leq x$.) The order is called *total* iff, for any $x, y \in S$, either $x \leq y$ or $y \leq x$. A totally ordered subset is also called a *chain.*

Let A be a nonempty subset of an ordered set S.

Any $x \in A$ is called a *maximal (minimal,* resp.) element of A iff $y \in A$ and $x \leq y$ ($y \leq x$, resp.) entail $x = y$.

Any $x \in A$ is called the *maximum (minimum,* resp.) of A iff $y \leq x$ ($x \leq y$, resp.) for any $y \in A$.

Any $x \in S$ is called an *upper (lower,* resp.) *bound* of A iff $y \leq x$ ($y \leq x$, resp.) for any $y \in A$.

Any $x \in S$ is called the *supremum* or *least upper bound (infimum* or *greatest lower bound,* resp.) of A iff it is the minimum (maximum, resp.) of the set of upper (lower, resp.) bounds of A. Such an element will be denoted by $\sup A$ ($\inf A$, resp.), if it exists.

S is called (superiorly) *inductive* iff any totally ordered (nonempty) subset has an upper bound.

S is called (superiorly) *completely inductive* iff any totally ordered (nonempty) subset has a supremum.

S is called a *lattice* iff any finite (nonempty) subset has a supremum and an infimum.

S is called a *complete lattice* iff any (nonempty) subset has a supremum and an infimum.

Theorem 2.1 (*Zorn's Lemma) Any nonempty inductively ordered set has a maximal element.* []

This classical result is equivalent to Zermelo's *axiom of choice* (which reads "the Cartesian product of any nonempty family of nonempty sets is nonempty"), and to the *theorem of well-ordering* ("any nonempty set S can be equipped with an order relation, such that any nonempty subset of S has a minimum element"). [] Zorn's Lemma is often applied through the next statement. [6]

Theorem 2.2 (*Hausdorff's Maximal Chain Theorem) Let S be a partially ordered set and $C \subset S$ be a chain (i.e., a totally order subset). Then there exists a chain $C^* \subset S$ that is maximal w.r.t. inclusion and such that $C \subset C^*$.*

In this statement we consider two ordered structures: that of S, and that by inclusion of the family $\mathcal{F}$ of the chains $\widetilde{C}$ such that $C \subset \widetilde{C} \subset S$. The maximality is referred to the latter structure.

[5] w.r.t. = with respect to.

[6] Zorn's lemma is often applied in alternative to the so-called *principle of transfinite induction:*
Let $(S, \leq)$ be a well-ordered set, and for any $x \in S$ let $P(x)$ be a statement such that
(i) $P(x)$ is true for the minimum element of S,
(ii) for any $x \in S$, if $P(y)$ for any $y \leq x$ with $y \neq x$, then $P(x)$ is true.
Then $P(x)$ is true for any $x \in S$.
(The argument is straightforward.)

Proof. Any chain $\mathcal{G} \subset \mathcal{F}$ is bounded above by (i.e., is included into) the union of all its elements, which indeed is an element of $\mathcal{F}$. Thus $\mathcal{F}$ is inductive with respect to the ordering by inclusion. It then suffices to apply Zorn's lemma. $\square$

2.2 Hahn-Banach theorem

This is just a reformulation of Theorem 2.29 of [Br].

Let X be a linear space over the field $\mathbb{K}$. A functional $p : X \to \mathbb{R}$ is called **sublinear** iff

$$p(\lambda_1 v_1 + \lambda_2 v_2) \le \lambda_1 p(v_1) + \lambda_2 p(v_2) \qquad \forall v_1, v_2 \in X, \forall \lambda_1, \lambda_2 \ge 0. \tag{2.1}$$

This holds iff p is subadditive and positively homogeneous (of degree 1), that is,

$$\begin{aligned} p(v_1 + v_2) \le p(v_1) + p(v_2) \qquad \forall v_1, v_2 \in X, \\ p(\lambda v) = \lambda p(v) \qquad \forall v \in X, \forall \lambda \ge 0, \end{aligned} \tag{2.2}$$

or equivalently p is convex and positively homogeneous. These functionals provide the natural environment for the results of this section.

Theorem 2.3 (*Hahn-Banach Theorem for real linear spaces) Let X be a real linear space, $p : X \to \mathbb{R}$ be a sublinear functional, M be a linear subspace of X, and $f : M \to \mathbb{R}$ be a linear functional such that $f(v) \le p(v)$ for any $v \in M$. Then there exists a linear functional $\widetilde{f} : X \to \mathbb{R}$ such that $\widetilde{f} = f$ in M and $\widetilde{f}(v) \le p(v)$ for any $v \in X$.* [7]

Proof. The set

$$\Phi := \big\{ g : \mathrm{Dom}(g) \to \mathbb{R} \,|\, M \subset \mathrm{Dom}(g) \subset X, g = f \text{ in } M, g \le p \big\}$$

can be partially ordered by setting

$$g_1 \preceq g_2 \quad \Leftrightarrow \quad \mathrm{Dom}(g_1) \subset \mathrm{Dom}(g_2), \;\; g_1 = g_2 \text{ in } \mathrm{Dom}(g_1).$$

We claim that this order is inductive, that is, any totally ordered subset has an upper bound. In fact, if $\{g_i\}_{i \in I}$ is a totally ordered subset of Φ, then, setting $\mathrm{Dom}(g) := \bigcup_i \mathrm{Dom}(g_i)$ and $g(u) = g_i(u)$ for any $u \in \mathrm{Dom}(g_i)$, we get $g \in \Phi$ and $g_i \preceq g$ for any i. By Zorn's Lemma 2.1 then there exists a maximal element $\widetilde{f} \in \Phi$. [8] At this point it suffices to prove that $\mathrm{Dom}(\widetilde{f}) = X$. By contradiction, let $u_0 \in X \setminus \mathrm{Dom}(\widetilde{f})$, and define a function h by setting

$$\begin{aligned} \mathrm{Dom}(h) := \{ v + \lambda u_0 : v \in \mathrm{Dom}(\widetilde{f}), \lambda \in \mathbb{R} \}, \\ h(v + \lambda u_0) := \widetilde{f}(v) + \lambda \alpha \qquad \forall v \in \mathrm{Dom}(\widetilde{f}), \forall \lambda \in \mathbb{R}. \end{aligned}$$

We claim that a real number α may be chosen in such a way that $h \le p$. This will entail that $\widetilde{f} \preceq h$, contradicting the maximality of $\widetilde{f}$, and will thus complete the proof.

By the positive homogeneity of p, it suffices to prove that

$$h(v + \lambda u_0) \le p(v + \lambda u_0) \qquad \forall v, w \in \mathrm{Dom}(\widetilde{f}), \lambda = \pm 1. \tag{2.3}$$

The linearity of $\widetilde{f}$, the inequality $\widetilde{f} \le p$ in $\mathrm{Dom}(\widetilde{f})$, and the subadditivity of p yield

$$\widetilde{f}(v) + \widetilde{f}(w) = \widetilde{f}(v + w) \le p(v + w) \le p(v - u_0) + p(w + u_0) \qquad \forall v, w \in \mathrm{Dom}(\widetilde{f}),$$

[7] When we drop any reference to the sublinear functional p in the theorem (and its proof), we obtain the standard extension theorem of linear functionals in linear spaces.

[8] Another argument may be based on Hausdorff's Maximal Chain Theorem, as in [Br]. Both procedures rest on *transfinite induction,* and are equivalent to Zermelo's axiom of choice.

whence

$$\widetilde{f}(v) - p(v - u_0) \leq p(w + u_0) - \widetilde{f}(w) \qquad \forall v, w \in \mathrm{Dom}(\widetilde{f}).$$

Therefore, for some $\alpha \in \mathbb{R}$,

$$\widetilde{f}(v) - p(v - u_0) \leq \alpha \leq p(w + u_0) - \widetilde{f}(w) \qquad \forall v, w \in \mathrm{Dom}(\widetilde{f}).$$

Defining the function h as above, this is tantamount to the inequality (2.3). □

Let us denote by $\Re(z)$ and $\Im(z)$ the real and the imaginary part of any complex number z, and notice that $\Im(z) = \Re(-iz) = -\Re(iz)$.

Lemma 2.4 *The real part g of any linear functional f on a linear space $V_{\mathbb{C}}$ over $\mathbb{C}$ is a linear functional on the associated linear space $V_{\mathbb{R}}$ over $\mathbb{R}$. Viceversa any linear functional g on $V_{\mathbb{R}}$ is the real part of the linear functional $f(v) = g(v) - ig(iv))$ on $V_{\mathbb{C}}$. Moreover, if $V_{\mathbb{C}}$ is a normed space, then f is continuous iff so is g. [Ex]*

Theorem 2.5 (*Hahn-Banach theorem for complex linear spaces)* [9] *Let X be a linear space over $\mathbb{K}$ equipped with a seminorm p, M be a linear subspace of X, and $f : M \to \mathbb{K}$ be a linear functional such that $|f(v)| \leq p(v)$ for any $v \in M$. Then there exists a linear functional $\widetilde{f} : X \to \mathbb{K}$ such that $\widetilde{f} = f$ in M and $|\widetilde{f}(v)| \leq p(v)$ for any $v \in X$.*

* *Proof.* If $\mathbb{K} = \mathbb{R}$ the statement directly follows from Theorem 2.3; let us then assume that $\mathbb{K} = \mathbb{C}$. Let us extend $\Re(f)$ to a linear functional $g : X \to \mathbb{R}$, as we did in Theorem 2.3; thus with $g \leq p$ in X. By the latter lemma the functional $\widetilde{f} : X \to \mathbb{C} : v \mapsto g(v) - ig(iv)$ is then linear and extends f.

For any fixed $v \in X$, we have $\widetilde{f}(v) = re^{i\theta}$ for some $r, \theta \geq 0$ (that may depend on v). Hence $\widetilde{f}(e^{-i\theta}v) = e^{-i\theta}\widetilde{f}(v) = r \geq 0$, and therefore $|\widetilde{f}(e^{-i\theta}v)| = \widetilde{f}(e^{-i\theta}v) = g(e^{-i\theta}v)$. By the positive homogeneity of p, we then have

$$|\widetilde{f}(v)| = |e^{-i\theta}\widetilde{f}(v)| = |\widetilde{f}(e^{-i\theta}v)| = g(e^{-i\theta}v) \leq p(e^{-i\theta}v) = |e^{-i\theta}|p(v) = p(v) \tag{2.4}$$

for all $v \in X$. □

*2.3 Some consequences of the Hahn-Banach theorem

The Hahn-Banach theorem has a number of relevant consequences in normed spaces.

By selecting any $f \not\equiv 0$ in the statement of the Hahn-Banach theorem, we get the next result.

Corollary 2.6 *The dual X' of any normed space $X \neq \{0\}$ is not reduced to the null functional.*

(This fails in L^p spaces for any $0 < p < 1$, and indeed these spaces cannot be equipped with any norm.) Thus *functional* analysis in normed spaces has nontrivial functionals at its disposal, whenever the underlying space X is nontrivial (i.e., $X \neq \{0\}$, as we shall systematically assume).

Corollary 2.7 *Let X be a normed space, M be a closed subspace of X, and $u \in X \setminus M$. Then there exists $f \in X'$ such that $f(u) = 1$ and $f(v) = 0$ for any $v \in M$. []*

We infer that the dual of a normed space separates points. That is, for any $x, y \in X$ such that $x \neq y$, there exists $f \in X'$ such that $f(x) \neq f(y)$. [Ex]

[9] Due to Bohnenblust-Sobczyk-Soukhomlinoff.

Corollary 2.8 *A linear subspace M of a normed space X is dense in X if (and only if) $f = 0$ is the only element of X' such that $f(v) = 0$ for any $v \in M$.* []

Corollary 2.9 *Let X be a normed space and M a linear subspace of X. Then any linear and continuous functional f on M can be extended to an $\widetilde{f} \in X'$ such that $\|\widetilde{f}\|_{X'} \leq \sup\{|f(v)| : v \in M\} =: |||f|||$. (Such an extension is said to be norm-preserving.)*

Proof. Let us apply the Hahn-Banach Theorem 2.3 with $p(v) = |||f|||\ \|v\|_X$. Thus there exists an extension $\widetilde{f} \in X'$ such that $|\widetilde{f}(v)| \leq |||f|||\ \|v\|_X$ for all $v \in X$; hence $\|\widetilde{f}\|_{X'} \leq |||f|||$. As $\widetilde{f}$ extends f, the latter is an equality. □

Corollary 2.10 *Let X be a real normed space. Then*

$$\|u\| = \max\{f(u) : f \in X', \|f\| \leq 1\} \qquad \forall\, u \in X. \tag{2.5}$$

In other terms, for any $u \in X$, there exists $f \in X'$ such that $\|f\| = 1$ and $f(u) = \|u\|$.

Proof. We have $f(u) \leq \|f\|_{X'}\|u\|_X \leq \|u\|$ whenever $\|f\| \leq 1$. In order to show the equality, let us fix an $u \in X$ and define the functional $g : M = \mathbb{K}u \to \mathbb{K}$ by setting $g(\lambda u) = \lambda\|u\|$. As $\|g\|_{M'} = 1$, by Corollary 2.9 there exists a norm-preserving extension $\widetilde{g}$. Thus $\|\widetilde{g}\| = 1$. □

2.4 Some classical separation results

(This subsection and the next one may replace Sect. 2.6 of [Br].)

Linear separation is an algebraic concept. We shall say that a set $H \subset X$ is a (linear) hyperplane iff $H = f^{-1}(0)$, $f : X \to \mathbb{R}$ ($f \not\equiv 0$) being a linear functional $f : X \to \mathbb{R}$; f is determined up to a factor $\lambda \neq 0$. We define an affine hyperplane as the translated of a hyperplane: $H = f^{-1}(0) + v$ for any f and $v \in X$; or equivalently, setting $\alpha = f(v)$, $H = f^{-1}(\alpha)$ for any f and $\alpha \in \mathbb{K}$. By an equivalent definition, a (linear) *hyperplane* is any proper linear subspace that is maximal, in the sense that X is the only linear subspace that strictly includes it. [10]

Let A, B be two nonempty subsets of a real linear space X. We say that a nonzero linear functional $f : X \to \mathbb{R}$ **separates** A and B iff $f(u) \leq f(v)$ for any $u \in A$ and any $v \in B$, that is,

$$\sup_{u \in A} f(u) \leq \inf_{v \in B} f(v). \tag{2.6}$$

Any $\alpha \in [\sup_A f, \inf_B f]$ defines an affine hyperplane $H = f^{-1}(\alpha)$ such that A and B lie on different sides of H. That is, A and B are respectively contained in the halfspaces $f^{-1}(]-\infty, \alpha])$ and $f^{-1}([\alpha, +\infty[)$. However, A and/or B might intersect H or even be contained into it.

We say that the linear functional f **strongly separates** [11] A and B iff

$$\sup_{u \in A} f(u) < \inf_{v \in B} f(v). \tag{2.7}$$

The notions of separation and of strong separation are trivially extended to normed spaces X assuming that $f \in X'$. In this case the affine hyperplane $H = f^{-1}(\alpha)$ is assumed to be closed. Notice that if M is a linear subspace, then the same holds for its closure, that however may have a larger dimension; for instance, a linear hyperplane $H = f^{-1}(0)$ is dense in X iff f is not continuous. Moreover $f \in X'$ separates two nonempty subsets A and B of X iff it separates their closures; the same property holds for the strong separation.

The following classical result is known as the **geometric form of the Hahn-Banach theorem.**

[10] After introducing the notion of codimension, we shall see that a hyperplane is a linear subspace of codimension 1.

[11] Here *strongly* does not refer to the strong topology!

* **Theorem 2.11** (*Ascoli-Mazur*) *Let X be a normed space over the field $\mathbb{K}$, M be an affine subspace [i.e., the translated of a linear subspace], and A be a nonempty open convex subset of X, such that $M \cap A = \emptyset$. Then there exists a closed affine hyperplane H such that $M \subset H$ and $H \cap A = \emptyset$. (In particular H thus separates M and A.)*

In other terms, we claim that there exist $f \in X'$ and $\alpha \in \mathbb{R}$ such that $f(v) = \alpha$ for any $v \in M$ and $f(v) > \alpha$ for any $v \in A$.

The requirement that A has interior points cannot be dispensed with, since an affine subspace may be dense. For instance, let us denote by c_{00} the set of the sequences that have just a finite number of nonvanishing terms. This is a proper dense vector subspace of $X = \ell^2$; so $f(c_{00}) = \mathbb{R}$ for any nonzero $f \in X'$, and consequently no point of $\ell^2 \setminus c_{00}$ can be separated from c_{00}.

Lemma 2.12 *Let A and B be two nonempty subsets of a normed space X. Then:*
(i) if A and B are convex, then $A + B$ is convex;
(ii) if A is open, then $A + B$ is open;
(iii) if A is compact and B is closed, then $A + B$ is closed.
(The properties that are here stated for $A + B$ also hold to $A - B := A + (-B)$.)

Proof. Part (i) and (ii) are straightforward. [Ex] Let us prove part (iii). For any point $w \in \overline{A + B}$, there exist sequences $\{u_n\} \subset A$ and $\{v_n\} \subset B$ such that $u_n + v_n \to w$. As A is compact, there exists a convergent subsequence $\{u_{n'}\}$ whose limit u belongs to A. Hence $v_{n'} \to v := w - u$, and $v \in B$ since B is closed. Thus $w = u + v \in A + B$. □

Even in $\mathbb{R}^2$, the set $A + B$ need not be closed if A and B are just closed. For instance, let $A_\pm$ be the graph of the real function $]0, +\infty[\to \mathbb{R} : x \mapsto \pm 1/x$, respectively. Then $(0, 0) \in \overline{A_+ + A_-}$ although $(0, 0) \notin A_+ + A_-$.

Theorem 2.13 (*Separation – Eidelheit*) *Let A and B be two disjoint nonempty convex subsets of a real normed space X, and A be open. Then A and B can be separated by a closed affine hyperplane.*

Proof. By Lemma 2.12 the set $A - B$ is convex and open. As $0 \notin A - B$, then by Theorem 2.11 the closed subspace $\{0\}$ can be separated from $A - B$; that is, there exists $f \in X'$ such that $f(A - B) \leq f(0) = 0$. As $f(A) - f(B) = f(A - B)$, we conclude that $f(A) \leq f(B)$. (One may also show that $f(A) < f(B)$.) □

Theorem 2.14 (*Strong Separation – Tukey and Klee*) *Let X be a real normed space, and A and C be two disjoint nonempty convex subsets of X, with A compact and C closed. Then A and C can be strongly separated by a closed affine hyperplane.*

Proof. As A is compact, $\varepsilon := \mathrm{dist}(A, C) = \inf \big\{ \|u - v\| : u \in A, v \in C \big\} > 0$. Thus, denoting by B_ε the open ball with radius ε centered in 0, the sets C and $A_\varepsilon = A + B_\varepsilon$ are disjoint. As by Lemma 2.12 A_ε is open and convex, by Theorem 2.13 there exists $f \in X'$ such $\sup f(A_\varepsilon) \leq \inf f(C)$. By choosing $u \in A$ with $f(u) = \max f(A)$ and $h \in X$ with $f(h) > 0$, we get $u + \delta h \in A_\varepsilon$ for a sufficiently small $\delta > 0$. Thus $\max f(A) = f(u) < f(u + \delta h) \leq \sup f(A_\varepsilon) \leq \inf f(C)$. □

Corollary 2.15 *Any closed convex subset A of a real normed space X is the intersection of the closed halfspaces that contain it.*

Proof. By Theorem 2.14, any point in the complement of A can be strongly separated from A, since singletons are compact. □

2.5 The Riesz characterization of finite-dimensional normed spaces

Let us first consider a simple example. Henceforth for any $p \in [1, +\infty]$ we shall denote by e_n the n**th unit vector** of ℓ^p, namely the sequence $(0, ..., 0, 1, 0, ...)$ with 1 at the nth place and 0 elsewhere.

No ball of ℓ^2 is compact: e.g., the sequence of the unit vectors $\{e_n\}$ is not of Cauchy, hence it has no convergent subsequence. The following classical result will allow us to extend this example to any infinite-dimensional Banach space. (This is nontrivial, as in generic Banach spaces there is no notion of orthogonality.)

Lemma 2.16 (*F. Riesz) Let M be a proper closed subspace of a normed space X over $\mathbb{K}$, and let $\theta < 1$. Then there exists $u \in X$ such that $\|u\| = 1$ and* $\inf\big\{\|u - v\| : v \in M\big\} \geq \theta$. [12]

Proof. There exists $u_1 \in X$ such that $\inf\big\{\|u_1 - v\| : v \in M\big\} \geq \theta$. For any $v_1 \in M$ such that $\|u_1 - v_1\| < 1$, the thesis is fulfilled by $u = (u_1 - v_1)/\|u_1 - v_1\|$. □

Theorem 2.17 (*F. Riesz) The closed unit ball of a normed space X is compact iff the space is finite-dimensional.*

Proof. In $X = \mathbb{K}^N$, and hence in any finite dimensional space, a set is compact iff it is closed and bounded. If X has infinite dimension, using the Riesz Lemma 2.16 we inductively construct a sequence $\{u_n\} \subset X$ such that $\|u_n\| = 1$ for any n and $\|u_n - u_m\| > 1/2$ whenever $n \neq m$. This sequence is not of Cauchy, hence it has no convergent subsequence. Thus the closed unit ball of X is not compact. □

As a consequence of this theorem, any compact subset of an infinite-dimensional normed space has empty interior.

*** Grothendieck's characterization of compact sets.** Bounded subsets of a finite-dimensional subspace of a Banach space are relatively compact. A relatively compact subset need not be finite-dimensional, but is not far from being so, because of the following nice result:

A subset K of a Banach space X is relatively compact iff there exists a vanishing sequence $\{u_n\}$ in X such that $K \subset \overline{\text{co}}\ (\{u_n\})$ (the closure of the convex hull of the elements of the sequence). []

Because of this result, if K is relatively compact then for any $\varepsilon > 0$ there exists $m \in \mathbb{N}$ such that

$$K \subset \overline{\text{co}}\ \big(\{u_1, ..., u_m\} \cup B(0, \varepsilon)\big). \qquad [Ex] \tag{2.8}$$

Thus, although compact subsets of a normed space need not be included into finite-dimensional subspaces, they are not far from that — loosely speaking.

2.6 Hamel and Schauder bases and dimension

A subset S of a linear space X is called an algebraic basis (or **Hamel basis**) if every element of X has a unique representation as a (finite) linear combination of elements of S. [13] This holds iff (i) S is linearly independent (i.e., any linear combination of its elements vanishes only if all its coefficients vanish), and (ii) S is maximal among all linearly independent subsets of X.

By the Hausdorff maximality principle, it is promptly seen that any linear space can be equipped with a Hamel basis, and that all of these bases have the same cardinality. [Ex] We can thus define the dimension of the space as the cardinality of any of its Hamel bases. (This is thus a purely algebraic notion.)

[12] The Riesz Lemma holds for $\theta = 1$ if M has finite dimension, but not in general.

[13] Without a notion of convergence, there is no natural way to define infinite linear combinations.

For any sequence $\{x_n\}$ in a normed space X, one says that the series $\sum_{n=1}^{\infty} x_n$ converges iff the sequence of its partial sums $\{S_m := \sum_{n=1}^{m} x_n\}_{m\in\mathbb{N}}$ converges. A sequence $\{x_n\}$ in X is called a topological basis (or a **Schauder basis)** iff every element of X has a unique representation of the form $\sum_{n=1}^{\infty} a_n x_n$, with $a_n \in \mathbb{K}$ for any n. This may depend on the order in which the elements x_n are enumerated. [] A Schauder basis is necessarily a linearly independent subset. If a normed space has a Schauder basis then it is separable. [14] [15]

2.7 * Algebraic complements and direct sums

The notion of projection somehow bridges linear, Banach and Hilbert spaces. First we address the algebraic side. Let M_1 and M_2 be two linear subspaces of a linear space V, and set

$$V = M_1 \oplus M_2 \quad \Leftrightarrow \quad M_1 + M_2 = V, \quad M_1 \cap M_2 = \{0\}. \tag{2.9}$$

This holds iff for any $x \in V$ there exists one and only one pair $(x_1, x_2) \in M_1 \times M_2$ such that $x = x_1 + x_2$. We then say that
V is the **algebraic direct sum** of M_1 and M_2, or that
M_1 and M_2 **algebraically complement** (or *supplement*) each other, or that
M_1 and M_2 are algebraically complemented (or supplemented).

This holds iff M_2 is (linearly) isomorphic to the linear quotient space V/M_1, or equivalently M_1 is isomorphic to the quotient space V/M_2:

$$V = M_1 \oplus M_2 \quad \Leftrightarrow \quad M_1 \sim V/M_2 \quad \Leftrightarrow \quad M_2 \sim V/M_1. \tag{2.10}$$

The dimension of V/M_1 is called the subspace **codimension** of M_1. For instance, a hyperplane is a linear subspace of codimension 1. Thus $V = M_1 \oplus M_2$ entails that

$$\text{codim } (M_1) = \dim \, (V/M_1) = \dim \, (M_2) = \text{codim } (V/M_2).$$

Setting $\infty + \infty = \infty$ and $\infty + n = \infty$ for any $n \in \mathbb{N}$, it is also promptly seen that

$$\dim \, (V) = \dim \, (M_1) + \dim \, (V/M_1),$$

but not necessarily $\dim \, (M_1) = \dim \, (V) - \dim \, (V/M_1)$. (Why?)

Any linear subspace A of a linear space V is algebraically complemented. (This may be proved via transfinite induction.) [Ex]

For any operator $L : V_1 \to V_2$ between linear spaces, let us set

$$\begin{aligned} &\mathcal{R}(L) := L(V_1) \ (\subset V_2) \quad \text{(range or image of } L\text{)}, \\ &\mathcal{N}(L) := L^{-1}(0) \ (\subset V_1) \quad \text{(kernel or nullspace of } L\text{)}. \end{aligned} \tag{2.11}$$

Proposition 2.18 *If V_1 and V_2 are linear spaces and $L : V_1 \to V_2$ is a linear operator, then*

$$\text{codim}(\mathcal{N}(L)) = \dim(\mathcal{R}(L)). \tag{2.12}$$

Proof. It suffices to notice that

$$V_1 = \mathcal{N}(L) \oplus V_1/\mathcal{N}(L) \qquad \text{(by an obvious identification)}, \tag{2.13}$$

and that the operator L induces a linear isomorphism between $V_1/\mathcal{N}(L)$ and $\mathcal{R}(L)$. □

[14] A topological space is called **separable** iff it has a countable dense subset. There is no relation between *separability, separatedness* (i.e., the Hausdorff property) and *separation* (between convex sets)!

[15] Surprisingly, there exist separable Banach spaces without any Schauder basis. [] This highly nontrivial result disproves a conjecture of Schauder himself.

In general there is no linear isomorphism between $V_1/\mathcal{R}(L)$ and $\mathcal{N}(L)$. Hence one cannot infer that $\dim(\mathcal{N}(L)) = \mathrm{codim}(\mathcal{R}(L))$. This equality may surely fail, since any V_2 may be replaced by a larger space.

2.8 * Projections in linear spaces

A linear operator $P : V \to V$ is called a **projection** on V iff it is **idempotent**, that is, $P^2 = P$ (or equivalently $P(I - P) = 0$).

Proposition 2.19 *Let V be a linear space. If P is a projection on V, then $\widetilde{P} := I - P$ is also a projection, and*

$$V = \mathcal{R}(P) \oplus \mathcal{N}(P) = \mathcal{N}(\widetilde{P}) \oplus \mathcal{R}(\widetilde{P}), \qquad P\widetilde{P} = \widetilde{P}P = 0. \tag{2.14}$$

Conversely, if M_1, M_2 are two linear subspaces of V and $V = M_1 \oplus M_2$, then there exists a unique pair of projections P and $\widetilde{P}$ on V such that $P + \widetilde{P} = I$ and

$$M_1 = \mathcal{R}(P) = \mathcal{N}(\widetilde{P}), \qquad M_2 = \mathcal{R}(\widetilde{P}) = \mathcal{N}(P), \qquad P\widetilde{P} = \widetilde{P}P = 0. \tag{2.15}$$

Proof. If P is a projection then obviously $I - P$ is a projection, too. For any $x \in V$, $x = P(x) + [x - P(x)]$ with $P(x) \in \mathcal{R}(P) = \mathcal{N}(\widetilde{P})$ and $x - P(x) \in \mathcal{R}(\widetilde{P}) = \mathcal{N}(P)$. Moreover if $x \in \mathcal{R}(P) \cap \mathcal{N}(P)$ then $x = P(x) = 0$. The first formula of (2.14) thus holds; the second one is trivial.

Let us now assume that $V = M_1 \oplus M_2$. For any $x \in V$ let (x_1, x_2) be the unique pair of $M_1 \times M_2$ such that $x = x_1 + x_2$, and set $P(x) := x_1$. It is straightforward to see that P is a projection on V, $M_1 = \mathcal{R}(P)$ and $M_2 = \mathcal{N}(P)$. By setting $\widetilde{P}(x) := x_2$ we then get (2.15).

Finally, it is clear that P is the unique projection such that $M_1 = \mathcal{R}(P)$ and $M_2 = \mathcal{N}(P)$. □

2.9 * Projections in Banach spaces

In normed spaces one is concerned with *continuous* projections.

Proposition 2.20 *Any projection P on a Banach space X is continuous iff both $\mathcal{R}(P)$ and $\mathcal{N}(P)$ are closed.* [16]

Proof. The "only if"-part is obvious, as $\mathcal{R}(P) = \mathcal{N}(I - P)$.

Let us come to the "if"-part. Let $u_n \to u$ and $Pu_n \to w$. As $Pu_n \in \mathcal{R}(P)$ and this set is closed, we infer that $w \in \mathcal{R}(P)$, whence $Pw = w$. Similarly, as $u - Pu_n \in \mathcal{N}(P)$ and this set is closed, we have $u - w \in \mathcal{N}(P)$, whence $Pu = Pw$. Thus $Pu = w$, that is, the graph of P is closed. It then suffice to apply the Closed Graph Theorem 5.10. □

If M_1, M_2 are closed subspaces of a Banach space X and $X = M_1 \oplus M_2$, then X is called a **topological direct sum**; in this case one says that M_1 and M_2 **topologically complement** (or *topologically supplement*) each other in X, and that they are topologically complemented. Notice that then M_2 is (isometrically) isomorphic to the quotient space X/M_1. Thus

$$X = M_1 \oplus X/M_1 \qquad \forall \text{ closed subspace } M_1 \text{ of } X. \tag{2.16}$$

At variance with what we saw in the purely algebraic setup,

$$\text{a closed subspace of a Banach space need not be topologically complemented,} \tag{2.17}$$

[16] This may be compared with the following statement: A linear functional $f : X \to \mathbb{R}$ is continuous if (and only if) $f^{-1}(0)$ is closed. [Ex]

i.e., it need not be either the range or the nullspace of any continuous projection. For instance, c_0 has no topological complement in ℓ^∞ (Phillips's theorem), [] although it has an algebraic complement. [17] However, a closed subspace of a Banach space is topologically complemented whenever either its dimension or its codimension are finite. [] [18]

2.10 Cartesian product of Banach spaces

For $i = 1, ..., M$, let X_i be a normed space over the field $\mathbb{K}$ (the same for all i). The Cartesian product $X_1 \times ... \times X_M$ may then be canonically equipped with the following norm $\| \cdot \|_1 := \sum_{i=1}^M \|\pi_i \cdot \|_i$, where by $\pi_i : X_1 \times ... \times X_M \to X_i$ we denote the ith canonic projection. That is,

$$\|u\|_1 = \sum_{i=1}^M \|u_i\|_{X_i} \qquad \forall u = (u_1, ..., u_M) \in X_1 \times ... \times X_M.$$

This norm is equivalent to any norm $\| \cdot \|_p := \left(\sum_{i=1}^M \|\pi_i \cdot \|_i^p \right)^{1/p}$ for any $p \in]1, +\infty[$, and also to any norm $\| \cdot \|_\infty := \max_{i=1,...,M} \|\pi_i \cdot \|_{X_i}$.

This may easily be extended to an infinite family of normed spaces (all over the same field). For instance, if i ranges in $\mathbb{N}$, then the following norm may be used:

$$\|u\|_1 = \sum_{i=1}^\infty \|u_i\|_{X_i} \qquad \forall u = (u_1, ..., u_i, ...) \in X_1 \times ... \times X_i \times ...$$

Here one may also introduce the analogous p-norms for any $p \in]1, +\infty]$, but these norms are not mutually equivalent (as it is obvious).

2.11 Exercises

1. = Is $\mathbf{Q}$ a Banach space? [19]

2. Let Ω be a Euclidean domain.
   (i) Is the linear space $C_b^0(\Omega)$ equipped with $\|\cdot\|_{L^\infty(\Omega)}$ a normed space? If so, is it complete?
   (ii) Is the linear space $C_c^0(\Omega)$ equipped with $\|\cdot\|_{L^2(\Omega)}$ a normed space? If so, is it complete?
   (iii) Is the linear space $C_b^1(\Omega)$ equipped with $\|\cdot\|_{C_b^0(\Omega)}$ a normed space? If so, is it complete?

3. (i) Is there a linear and continuous operator L from a noncomplete normed space X to a Banach space Y, such that $L(X)$ is infinite dimensional?
   (ii) Is there a linear and continuous operator L from a Banach space X to a noncomplete normed space Y, such that $L(X)$ is infinite dimensional?

4. (i) Is the operator $L : c_{00} \to c_{00} : \{u_n\} \mapsto \{nu_n\}$ bounded?
   (ii) Exhibit a sequence $\{u_j\}$ such that $u_j \to 0$ but $Lu_j \not\to 0$.

[17] Let us set $C_0^0([0,1]) := \{v \in C^0([0,1]) : v(0) = v(1) = 0\}$, $C_0^0(\mathbb{R}) := \{v \in C^0(\mathbb{R}) : v(x) \to 0 \text{ as } x \to \pm\infty\}$, $C_b^0(A) := \{v : A \to \mathbb{C} \text{ continuous and bounded}\}$ for any $A \subset \mathbb{R}$, and equip these spaces with the uniform norm. Note the analogies:

$$c_{00} \text{ is dense in } c_0; \quad c_0, c \text{ are closed subspaces of } \ell^\infty; \tag{2.18}$$

$$C_c^0(\mathbb{R}) \text{ is dense in } C_0^0(\mathbb{R}); \quad C_0^0(\mathbb{R}), C_b^0(\mathbb{R}) \text{ are closed subspaces of } L^\infty(\mathbb{R}); \tag{2.19}$$

$$C_c^0(]0,1[) \text{ is dense in } C_0^0([0,1]); \quad C_0^0([0,1]), C_b^0([0,1]) \text{ are closed subspaces of } L^\infty(0,1). \tag{2.20}$$

[18] A Banach space X is topologically isomorphic to a Hilbert space iff any closed subspace of X is topologically complemented (Lindenstrauss-Tzafriri's Theorem). []

[19] the symbol "=" at the beginning of the text is not for the reader.

5. (i) Is c_{00} a closed subspace of ℓ^p for some $p \in [1, +\infty]$?
(ii) Is c_{00} dense in ℓ^p for any $p \in [1, +\infty]$?
(iii) Is ℓ^p dense in ℓ^q for any exponent $1 \le p, q \le +\infty$?

6. = Let $\{u_n\}_{n \in \mathbb{N}} \subset c_{00}$, and use the notation $u_n = ((u_n)_1, (u_n)_2, ..., (u_n)_j, ...)$ for any n. Assume that $(u_n)_j \to 0$ as $n \to +\infty$, for any $j \in \mathbb{N}$. Does then $u_n \to 0$ in ℓ^∞?

7. = Let Ω be a Euclidean set, and let the space $B(\Omega)$ of bounded $\mathbb{K}$-valued functions be equipped with the uniform norm (i.e., the sup-norm, not the essential-sup norm !).

(i) Is $B(\Omega)$ a normed space? is it complete?
* (ii) Is the linear space of bounded representatives of $L^\infty(\Omega)$ dense in $B(\Omega)$?

8. Establish whether the following functionals $C^1([0,1]) \to \mathbb{R}$ are continuous, and whether they can be extended to continuous functionals $C^0([0,1]) \to \mathbb{R}$:

(i) $u \mapsto u'(0)$; (ii) $u \mapsto \int_0^1 u(1-x)\,dx$; (iii) $u \mapsto \int_0^1 u(e^{x/4})\,dx$.

9. = Let us equip $X := \{v \in C^0([0,1]) : v(0) = 0\}$ with the uniform norm. Notice that $f : v \mapsto \int_0^1 v(x)\,dx$ is an element of X'.

(i) Is X a Banach space?
(ii) Is X a closed subspace of $C^0([0,1])$?
(iii) Does $u \in X$ with $\|u\|_X = 1$ exist such that $f(u) = \|f\|_{X'}$?

10. (i) In the Riesz Lemma 2.16 may the closedness of M be dropped?
(ii) Show that this lemma holds with $\theta = 1$ if M is finite dimensional.

11. = * (i) Is $\{v\big|_{]0,1[} : v \in C^0([0,1])\}$ a dense subspace of $C^0_b(]0,1[)$?
* (ii) Is $C^1_b(\mathbb{R})$ dense in $C^0_b(\mathbb{R})$?

12. = * May every $L \in \mathcal{L}(c_{00}, c_{00})$ be extended to an operator $\widetilde{L} \in \mathcal{L}(c_0, c_{00})$?

Hint: Consider the identity operator...

13. (i) For which $p \in [1, +\infty]$ is ℓ^p separable?
(ii) Is $L^2(0,1)$ separable?
(iii) Is $L^2(\mathbb{R})$ separable?

14. = (i) Is c_0 separable?
(ii) Is c separable?
(iii) Is $C^0([0,1])$ separable?
* (iv) Is $C^0_b(\mathbb{R})$ separable?
* (v) Is $C^0_b([0,1[)$ separable?

15. Show that the functional $f : c \to \mathbb{K} : \{u_n\} \mapsto \lim_n u_n$ is linear and continuous, and calculate its norm.

16. = (i) Is c isomorphic to ℓ^∞?
* (ii) Is c isomorphic to c_0?

17. = Let X be a Banach space and $U \subset X$.

(i) Check that in general $U + U \neq 2U$, and prove that $U + U = 2U$ if U is convex.
* (ii) Show by a counterexample that if $U + U = 2U$ then U need not be convex.

18. (i) Let A and B be two nonempty compact subsets of a normed space X. Prove that $A+B$ is compact.
(ii) Show by a counterexample that A compact and B closed do not entail that $A + B$ is compact.

19. (i) Is it true that two nonempty convex subsets A and B of a real linear space can be separated iff $A - B$ can be separated from $\{0\}$?
(ii) Does this hold for the strong separation, too?

20. * A series $\sum_{n=1}^{\infty} u_n$ in a normed space X is called **convergent** iff the sequence formed by its partial sums $\{S_m = \sum_{n=1}^{m} u_n\}$ converges in X. The series is called **totally** (or **absolutely) convergent** iff the series $\sum_{n=1}^{\infty} \|u_n\|$ converges (in $\mathbb{K}$), that is, the sequence of the norms is an element of ℓ^1.

    Prove that a normed space X is complete iff any totally convergent series in X is convergent.

    Hint for the "if"-part: Let $\{u_j\}$ be a Cauchy sequence in X, and recursively construct a subsequence $\{u_{n_j}\}$ as follows: for any j, n_j is selected so that $\|u_\ell - u_m\| \le 2^{-j}$ for any $\ell, m > n_j$. Therefore $\|u_{n_{j+2}} - u_{n_{j+1}}\| \le 2^{-j}$ for any j, whence $\sum_{j=1}^{\infty} \|u_{n_{j+1}} - u_{n_j}\| < +\infty$. The subsequence $\{u_{n_j}\}$ then converges...

21. = * Give an example of a totally convergent series in c_{00} that does not converge.

22. In Banach spaces are all convergent series totally convergent?

23. = * (i) Show that the dimension of a Banach space cannot be countable.
    Hint: Use the Baire theorem...
    * (ii) Give an example of a normed space that has countable infinite dimension.

24. = Let X be the set of the real sequences $\{u_n\}$ such that $u_n = o(n^{-1/2})$ as $n \to +\infty$.

    (i) Is $X \subset \ell^p$ for some p?
    (ii) Is $X \supset \ell^p$ for some p?

25. = Let X be the set of the measurable functions $v : \mathbb{R}^+ \to \mathbb{R}$ such that $v(x) = o(x^{-1/2})$ as $x \to +\infty$.

    (i) Is $X \subset L^p(\mathbb{R}^+)$ for some p?
    (ii) Is $X \supset L^p(\mathbb{R}^+)$ for some p?

26. = For any $n \in \mathbb{N}$, let $a_n > 0$ and χ_n be the characteristic function of the interval $[n, n+a_n]$.

    (i) Do sequences $\{a_n\}$ exist such that the sequence $\{\chi_n\}$ converges in $L^p(\mathbb{R})$ for some p? Must $\{a_n\}$ be related to p? What is the limit?
    (ii) Do sequences $\{a_n\}$ exist such that the set $\{\chi_n\}$ is relatively sequentially weakly star compact in $L^p(\mathbb{R})$ for some p?
    (iii) Do sequences $\{a_n\}$ exist such that $\{\chi_n\}$ weakly converges in $L^p(\mathbb{R})$ for some $p \neq \infty$? What is the limit?
    (iv) Do sequences $\{a_n\}$ exist such that $\{\chi_n\}$ weakly star converges in $C_c^0(\mathbb{R})'$? What is the limit?

27. = Do the following sequences converge in some sense in $L^p(-1,1)$ for some $p \in \,]1, +\infty]$? What is the limit?

    (i) $\{u_n(x) := e^{x/n} \cos(nx)\}$;
    (ii) $\{u_n(x) := [\sin(nx)]^+\}$;

28. Let us equip $C_c^0(\mathbb{R})$ with the uniform norm.

    (i) Is $C_c^0(\mathbb{R})$ a closed subspace of $L^\infty(\mathbb{R})$?
    (ii) Is $C_c^0(\mathbb{R})$ dense in $C_b^0(\mathbb{R})$?
    (iii) Is $C_c^0(\mathbb{R})$ dense in $L^1(\mathbb{R})$?

29. = Let X be a normed space, and $\{u_n\}, \{f_n\}, \{F_n\}$ be sequences in X, X', X'', respectively. Are the following implications true?

    (i) if $u_n \to u$ weakly and $f_n \to f$ strongly, then $\langle f_n, u_n \rangle \to \langle f, u \rangle$;
    (ii) if $u_n \to u$ strongly and $f_n \to f$ weakly, then $\langle f_n, u_n \rangle \to \langle f, u \rangle$;
    (iii) if $u_n \to u$ strongly and $f_n \to f$ weakly star, then $\langle f_n, u_n \rangle \to \langle f, u \rangle$;
    (iv) if $u_n \to u$ weakly and $f_n \to f$ weakly star, then $\langle f_n, u_n \rangle \to \langle f, u \rangle$;

    (v) if $F_n \to F$ weakly and $f_n \to f$ strongly, then $\langle F_n, f_n \rangle \to \langle F, f \rangle$;
    (vi) if $F_n \to F$ strongly and $f_n \to f$ weakly, then $\langle F_n, f_n \rangle \to \langle F, f \rangle$;

(vii) if $F_n \to F$ strongly and $f_n \to f$ weakly star, then $\langle F_n, f_n \rangle \to \langle F, f \rangle$;
(viii) if $F_n \to F$ weakly star and $f_n \to f$ strongly, then $\langle F_n, f_n \rangle \to \langle F, f \rangle$;
(ix) if $F_n \to F$ weakly and $f_n \to f$ weakly, then $\langle F_n, f_n \rangle \to \langle F, f \rangle$.

30. = This exercise characterizes the weak convergence by *testing* it on a subfamily of the dual space.

(i) Let $\{u_n\}$ be a bounded sequence in a normed space X, $u \in X$, and A be a dense subset of the unit sphere of X'. Prove that $u_n \to u$ weakly whenever $\langle f, u_n \rangle \to \langle f, u \rangle$ for any $f \in A$.
(ii) Characterize similarly the weak convergence in X'.
(iii) Characterize similarly the weak star convergence in X'.

31. = ** Let X be a normed space. Show that a linear functional $f : X \to \mathbb{R}$ is continuous if (and only if) its kernel $f^{-1}(0)$ is closed.

Hint: If $f \in X' \setminus \{0\}$ then there exists $\bar{v} \in X$ such that $f(\bar{v}) = 1$. For any closed $A \subset \mathbb{R}$ then $f^{-1}(A) = A\bar{v} + f^{-1}(0)$ $(= \{a\bar{v} + w : a \in A, w \in f^{-1}(0)\})$

32. = May two nonisomorphic normed spaces X_1 and X_2 be such that $X_1 \subset X_2$ (setwise) and $\|v\|_1 \leq \|v\|_2$ for any $v \in X_1$?

33. = * Let A be a linear subspace of a Banach space X.

(i) Give a sufficient condition such that $\mathcal{N}(L) = A$ for some $L \in \mathcal{L}(X)$.
(ii) Give a necessary and sufficient condition such that $A = f^{-1}(0)$ for some $f \in X'$ with $f \not\equiv 0$.

34. = * Let X, Y, Z be Banach spaces, and $L \in \mathcal{L}(X, Y)$, $M \in \mathcal{L}(Y, Z)$. If ML has a linear and continuous inverse, does then follow that both M and L have a linear and continuous inverses?

35. = * Let X, Y be Banach spaces, and $L : X \to Y$ be a linear mapping.
(i) Prove that, if $u_n \to u$ entails $Lu_n \rightharpoonup Lu$, then L is continuous (that is, $u_n \to u$ entails $Lu_n \to u$).

Hint: Use the closed graph theorem ...
(ii) If Y has a predual, does this result hold also assuming that $u_n \to u$ entails $Lu_n \overset{*}{\rightharpoonup} u$?

36. Let X be the space of compactly-supported continuous functions $]0, 1] \to \mathbb{C}$; let Y be the space of continuous functions $]0, 1] \to \mathbb{C}$ that vanish as $t \to 0$; equip both spaces with the sup-norm.

(i) are they Banach spaces? are they Fréchet spaces?
(ii) are they normed subspaces of $L^2(0, 1)$? $L^2(0, 1)$? of $L^1(0, 1)$?
* (iii) Is X dense in Y?

3 Spaces of Continuous Functions

*3.1 Banach algebras

A vector space A over the field $\mathbb{K}$ $(= \mathbb{R}$ or $\mathbb{C})$ is called an (associative) **algebra** iff it is equipped with a multiplication $A \times A \to A$ such that

$$a(bc) = (ab)(c)\,,$$
$$(a + b)c = ac + bc, \qquad a(b + c) = ab + ac\,,$$
$$\lambda(ab) = (\lambda a)b = a(\lambda b)$$

for any $a, b, c \in A$ and any $\lambda \in \mathbb{K}$. The algebra is called **commutative** iff $ab = ba$ for any $a, b \in A$. If A is an algebra as well as a Banach space over $\mathbb{K}$, it is called a **Banach algebra** iff

$$\|ab\| \leq \|a\| \, \|b\|$$

for any $a, b \in A$. This entails that the multiplication is continuous on $A \times A$. If moreover A has a multiplicative unit I with $\|I\| = 1$, it is called a **Banach algebra with unit**.

Examples of Banach algebras.
(i) The field $\mathbb{K}$ is a trivial commutative Banach algebra with unit. For any $N > 1$, $\mathbb{K}^{N\times N}$ is a noncommutative Banach algebra with unit (the unit matrix).
(ii) The space $\mathcal{L}(X)$ of bounded linear operators over a Banach space X, equipped with the composition, is a Banach algebra with unit (the identity operator). If X has finite dimension, this algebra is isomorphic to $\mathbb{K}^{N\times N}$.
(iii) For any topological space T, let us equip the Banach space $C^0_b(T)$ of bounded continuous functions $T \to \mathbb{K}$ with the supremum norm and the pointwise multiplication. This is a commutative Banach algebra with unit (the constant function 1).
(iv) For any measure space $(\Omega, \mathcal{A}, \mu)$, the space $L^\infty(\Omega)$, equipped with the pointwise multiplication is a commutative Banach algebra with unit (the constant function 1).
(v) For any $N > 1$, $C^0_b(T; \mathbb{K}^{N\times N})$ and $L^\infty(\Omega; \mathbb{K}^{N\times N})$ are noncommutative Banach algebras with unit.

Theorem 3.1 (*Neumann Series) Let X be a Banach algebra with unit I. For any $u \in X$,*

$$\|u\| < 1 \quad \Rightarrow \quad \exists (I - u)^{-1} = \sum_{n=0}^{\infty} u^n. \tag{3.1}$$

Proof. It suffices to notice that if $\|u\| < 1$ then

$$(I - u) \sum_{n=0}^{\infty} u^n = \left(\sum_{n=0}^{\infty} u^n \right) (I - u). = I \qquad \square$$

3.2 The Ascoli-Arzelà theorem

The following classical result conveys an important characterization of (relative) compactness in $C^0(K)$ (K being a compact metric space). This is relevant since the space $C^0(K)$ has no predual, and thus here one cannot use the Banach-Alaoglu theorem.

Theorem 3.2 (*Ascoli-Arzelà) Let K be a compact metric space. A subset $\mathcal{F}$ of $C^0(K)$ is relatively strongly compact if (and only if) it is (equi)bounded as well as uniformly equicontinuous in $C^0(K)$, that is,* [20]

$$\sup \left\{ |u(x)| : x \in K, u \in \mathcal{F} \right\} < +\infty, \tag{3.2}$$

$$\sup \left\{ |u(x) - u(y)| : x, y \in K, d(x, y) \leq h, u \in \mathcal{F} \right\} \to 0 \qquad \text{as } h \to 0. \tag{3.3}$$

[20] Because of the compactness of K, the condition (3.3) is equivalent to the (in general weaker) condition of pointwise equicontinuity

$$\sup_{u \in \mathcal{F}} \sup \left\{ |u(x) - u(y)| : y \in K, d(x, y) \leq h, u \in \mathcal{F} \right\} \to 0 \qquad \text{as } h \to 0, \forall x \in K. \quad [Ex]$$

**Proof.* It suffices to show that a Cauchy subsequence may be extracted from any sequence in $\mathcal{F}$. Let $\{x_j\}_{j\in\mathbb{N}}$ be a (countable) dense subset in K. Because of the boundedness of $\mathcal{F}$, from $\{u_n\}$ one may extract a subsequence $\{u_{n_1}\}$ such that $\{u_{n_1}(x_1)\}$ converges. Similarly, for $j = 2, 3, ...$, from $\{u_{n_{j-1}}\}$ one may iteratively extract a subsequence $\{u_{n_j}\}$ in such a way that $\{u_{n_j}(x_\ell)\}$ converges for all $\ell \leq j$. By a diagonalization procedure, for any $m \in \mathbb{N}$ let us then define $\widetilde{u}_m$ as the mth element of the sequence u_{n_m}. Thus $\{\widetilde{u}_m\}$ is a subsequence extracted not only from the initial sequence $\{u_n\}$ but also from $\{u_{n_j}\}$ for any $j \in \mathbb{N}$; moreover $\{\widetilde{u}_m(x_j)\}$ converges for any $j \in \mathbb{N}$.

Let us now fix any $\varepsilon > 0$. By equicontinuity there exists a $\delta > 0$ such that

$$|\widetilde{u}_m(x_j) - \widetilde{u}_m(y)| \leq \varepsilon \qquad \forall y \in K \cap B(x_j, \delta), \forall j, m \in \mathbb{N}. \tag{3.4}$$

By the compactness of K, a finite subcovering $\{B(x_j, \delta)\}_{j\in J}$ may be extracted from the family of the open balls $\{B(x_j, \delta)\}_{j\in\mathbb{N}}$. Therefore, for any m', m'' large enough,

$$\begin{aligned} &|\widetilde{u}_{m'}(y) - \widetilde{u}_{m''}(y)| \\ &\leq |\widetilde{u}_{m'}(y) - \widetilde{u}_{m'}(x_j)| + |\widetilde{u}_{m'}(x_j) - \widetilde{u}_{m''}(x_j)| + |\widetilde{u}_{m''}(x_j) - \widetilde{u}_{m''}(y)| \\ &\overset{(3.4)}{\leq} |\widetilde{u}_{m'}(x_j) - \widetilde{u}_{m''}(x_j)| + 2\varepsilon \qquad \forall y \in K \cap B(x_j, \delta), \forall j \in J, \end{aligned}$$

whence

$$\max_{y\in K} |\widetilde{u}_{m'}(y) - \widetilde{u}_{m''}(y)| \leq \max_{j\in A} |\widetilde{u}_{m'}(x_j) - \widetilde{u}_{m''}(x_j)| + 2\varepsilon.$$

As $\{\widetilde{u}_m(x_j)\}$ is a Cauchy sequence (in $\mathbb{R}$) for any j, $\{\widetilde{u}_m\}$ is then a Cauchy sequence in $C^0(K)$. □

3.3 Exercises

1. Drop any of the three assumptions of the Stone-Weierstrass theorem (1. separation, 2. constants, 3. conjugation), and find a counterexample in each case.

2. = Does the Weierstrass theorem hold in the algebra $C^0(]0, 1[)$?

3. = Discuss the mutual implications among the following notions
   (i) pointwise equicontinuity of a family of functions in $C^0_b(\mathbb{R})$;
   (ii) equilipschitzianity of a family of functions in $C^0_b(\mathbb{R})$ (i.e., equiboundedness of the incremental ratios);
   (iii) equiboundedness of a family of functions in $C^1_b(\mathbb{R})$.
   (iv) uniform equicontinuity of a family of functions in $C^0_b(\mathbb{R})$;

4. Let $\{u_n\}$ be an equibounded, and uniformly equicontinuous family of monotone functions in $C^1_b(\mathbb{R})$. Does the pointwise convergence of $\{u_n\}$ to 0 entail the uniform convergence of a subsequence?

5. Is the proof of Lemma 3.11 at p. 54 of [Br] correct?

6. * (i) Prove that any pointwise equicontinuous sequence of functions in $C^0(K)$ is uniformly equicontinuous, assuming that K is a compact metric space.

   (ii) Exhibit a counterexample for a noncompact K.

4 * Weak Topologies

(Yes: this is the asterisk, not the star of the weak star topology. More material is available on demand.)

4.1 * Final topology

Before introducing weak topologies, we review two standard methods of constructing a topology on a given set, in such a way that certain continuity properties are fulfilled.

Let A be any index set. Let $\{X_\alpha\}_{\alpha\in A}$ be a family of topological spaces, and $\{\lambda_\alpha\}_{\alpha\in A}$ be a family of mappings $X_\alpha \to X$, where X is any set. Let us say that a subset U of X is open iff $\lambda_\alpha^{-1}(U)$ is open for all $\alpha \in A$. This defines a topology on X which is called the **final** (or *inductive*) topology on X generated by the family $\{(X_\alpha, \lambda_\alpha)\}_{\alpha\in A}$. This is the finest among the topologies on X that make all λ_α's continuous.

Proposition 4.1 *Let X be equipped with the final topology generated by a family $\{(X_\alpha, \lambda_\alpha)\}_{\alpha\in A}$ as above. Then:*

(i) For any topological space H, a mapping $f : X \to H$ is continuous iff $f \circ \lambda_\alpha : X_\alpha \to H$ is continuous for any $\alpha \in A$.

(ii) The property (i) characterizes the final topology.

Proof. (i) By definition, for any open subset V of H, $f^{-1}(V)$ is open in X iff $(f \circ \lambda_\alpha)^{-1}(V) = \lambda_\alpha^{-1}(f^{-1}(V))$ is open in X_α for any α. This yields the first statement.

(ii) Let us now denote by $\hat{X}$ the set X equipped with another topology which fulfills the property (i). By applying the "if" part of (i) to the identity mapping $j : X \to \hat{X}$ and to $j^{-1} : \hat{X} \to X$, we see that j is a homeomorphism. □

*4.2 Initial topology

The following construction may be regarded as dual to the previous one.

Let A still be any index set. Let us consider a set Y, a family $\{Y_\alpha\}_{\alpha\in A}$ of topological spaces, and a family $\{\gamma_\alpha\}_{\alpha\in A}$ of mappings $Y \to Y_\alpha$. Let the family $\mathcal{S}$ of subsets of Y be defined as

$$\mathcal{S} = \{\gamma_\alpha^{-1}(V_\alpha) : \ \alpha \in A, \, V_\alpha \text{ open in } Y_\alpha\}, \tag{4.1}$$

define U to be open in Y iff it is a union of finite intersections [21] of sets in $\mathcal{S}$. The resulting topology is called the **initial** (or *projective*) topology on Y generated by the family $\{(Y_\alpha, \gamma_\alpha)\}_{\alpha\in A}$. It is the coarsest topology on X under which any γ_α is continuous.

Proposition 4.2 *Let Y be equipped with the initial topology generated by a family $\{(Y_\alpha, \gamma_\alpha)\}_{\alpha\in A}$ as above. Then:*

(i) For any topological space H, a mapping $f : H \to Y$ is continuous iff $\gamma_\alpha \circ f : H \to Y_\alpha$ is continuous for any $\alpha \in A$.

(ii) The property (i) characterizes the initial topology.

Proof. (i) For any α and any open subset V_α of Y_α, $\gamma_\alpha^{-1}(V_\alpha)$ is open in Y iff $(\gamma_\alpha \circ f)^{-1}(V_\alpha) = f^{-1}(\gamma_\alpha^{-1}(V_\alpha))$ is open in H. This yields the first statement.

(ii) Let us now denote by $\hat{Y}$ the set Y equipped with another topology which fulfils (i). By applying the "if" part of (i) to the identity mapping $j : Y \to \hat{Y}$ and to $j^{-1} : \hat{Y} \to Y$, we see that j is a homeomorphism. □

For instance, the Cartesian product $Y := \prod_{\alpha\in A} Y_\alpha$ of a family of topological spaces $\{Y_\alpha\}$ may be equipped with the initial topology generated by the family of the canonical projections

[21] In topological terminology, $\mathcal{S}$ is a **subbasis** of that topology.

$\pi_\alpha : Y \to Y_\alpha$. This is called the *product topology.* The topology induced by a topological space T on a subset M is another example of an initial topology. In this case $Y = M$, the index set A is a singleton, $Y_\alpha = T$, and $\gamma_\alpha : M \to T$ is the canonical imbedding.

*4.3 Construction of weak topologies

In any normed space X, besides the topology generated by the norm (usually referred to as the **strong topology**), a **weak topology** is defined. This is the initial topology on X generated by the family $\{(\mathbb{K}, f) : f \in X'\}$. Thus

$$\begin{array}{c}\text{the weak topology on } X \text{ is the coarsest topology on } X \\ \text{among those that make all functionals of } X' \text{ continuous.}\end{array} \tag{4.2}$$

As a consequence, any weakly closed (weakly open, resp.) set is closed (open, resp.). [22] A subbasis $\mathcal{S}$ of the weak topology in X is given by

$$\mathcal{S} = \bigcup_{u \in X} (u + \mathcal{S}_0), \quad \mathcal{S}_0 = \{f^{-1}(]-\varepsilon, \varepsilon[) : f \in X', \varepsilon > 0\}. \tag{4.3}$$

The family $\mathcal{S}_0$ is thus a subbasis of the system of neigborhoods of the origin. Any weakly open subset of X is the union of a family of elements, each being the intersections of a finite subfamily of $\mathcal{S}$. The same construction applies to the dual space X'. Thus

$$\begin{array}{c}\text{the weak topology on } X' \text{ is the coarsest topology on } X' \\ \text{among those that make all functionals of } X'' \text{ continuous.}\end{array} \tag{4.4}$$

In X' the **weak star topology** is also defined. [23] This is the initial topology on X' generated by the family

$$\mathcal{F} := \{(\mathbb{K}, F) : F \in J(X)\} = \{(\mathbb{K}, J_u) : u \in X\},$$

where by J we denote the canonical imbedding $X \to X''$. Thus

$$\begin{array}{c}\text{the weak star topology on } X' \text{ is the coarsest topology on } X' \\ \text{among those that make all functionals of } J(X) \text{ continuous.}\end{array} \tag{4.5}$$

If X is reflexive, i.e. $J(X) = X''$, then the weak star topology coincides with the weak topology on X'.

*4.4 Schur and von Neumann

The weak topology is not metrizable whenever X has infinite dimension. Therefore the weak closure of a subset S of X may include elements that cannot be represented as the weak limit of any sequence in S. Therefore weak continuity and sequential weak continuity need not coincide. Nevertheless,

$$\text{a sequence in } \ell^1 \text{ converges weakly iff it converges strongly.} \tag{4.6}$$

This astonishing phenomenon is known as the Schur property. For the reasons that we just explained, this does not contradict the next result.

[22] The norm topology will be our default topology unless otherwise specified. So open will stand for strongly open, closed for strongly closed, and so on.

[23] This is meaningful only in a dual space. The *star* refers to the fact that in the literature the dual space is often denoted by X^*, with a star instead of a prime.

Theorem 4.3 (*Banach) The weak and the strong topology coincide only in finite-dimensional normed spaces.*

The next example is due to von Neumann (who introduced the notion of weak topology). In ℓ^2, for any n let us denote by e_n the nth vector, and define the (unbounded) set $A := \{e_m + me_n : m, n \in \mathbb{N}, 1 \leq m < n\}$. No sequence of A weakly vanishes in ℓ^2. [Ex] Nevertheless the null element $\underline{0} := (0, ..., 0, ...)$ is in the weak closure of A, since any weak neighborhood of $\underline{0}$ intersects A. []

*4.5 Uniqueness of weak and weak star limits

We recall the reader that one says that two points u, v in a topological space H can be **separated** whenever there exist two disjoint open sets that respectively include u and v. The space H is called a **Hausdorff space**, and its topology is said to be **Hausdorff**, if any pair of distinct points $u, v \in H$ can be separated. This holds iff the limit of any converging sequence in H is unique.

The next result thus entails the uniqueness of weak and weak star limits.

Proposition 4.4 *The weak topology of a normed space X and the weak star topology of X' are Hausdorff.*

Proof. Let us first consider the weak topology. If $u, v \in X$ with $u \neq v$, then $f(u) \neq f(v)$ for some $f \in X'$ by separation (Theorem 2.14), so $f^{-1}(I)$ and $f^{-1}(J)$ separate u and v, where I and J are any open real intervals which separate $f(u)$ and $f(v)$.

Let us next come to the weak star topology. Just by definition of function, without using any consequence of the Hahn-Banach theorem, if $f, g \in X'$ with $f \neq g$, then $f(u) \neq g(u)$ for some $u \in X$: The argument then proceeds as above. □

* If X is an infinite dimensional normed space, X' equipped with the weak topology is not complete. []

*4.6 Two classical results of Mazur

A function $f : X \to \mathbb{R} \cup \{+\infty\}$ is said to be **lower semicontinuous** (**convex**, resp.) iff for any $a \in \mathbb{R}$ its **epigraph** $\{(v, r) \in X \times \mathbb{R} : v \in X, f(v) \leq r\}$ is closed (convex, resp.).

Theorem 4.5 (*Mazur) Let X be a real normed space. Then:*
(i) A convex subset of X is closed iff it is weakly closed.
(ii) A convex function $f : X \to \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous iff it is weakly lower semicontinuous.
(iii) A linear functional $f : X \to \mathbb{R}$ is continuous iff it is weakly continuous.

Proof. After Corollary 2.15, any closed convex subset A of X is the intersection of the closed halfspaces that contain it; moreover any half space is closed iff it is weakly closed. This yields part (i). Part (ii) directly follows from part (i). Finally, part (ii) entails part (iii), since for any linear functional f both f and $-f$ are convex. □

Remarks. (i) More generally, if X and Y are Banach space and $L : X \to Y$ is a linear operator, then each of the following properties is equivalent to the other ones:
(1) $L : X \to Y$ is continuous from X_{strong} to Y_{strong} (i.e., L is "strongly continuous"),
(2) $L : X \to Y$ is continuous from X_{strong} to Y_{weak},
(3) $L : X \to Y$ is continuous from X_{weak} to Y_{weak} (i.e., L is "weakly continuous").

Here the spaces are assumed to complete, as the proof rests on the closed graph theorem. []

(ii) If X has a predual (i.e., there exists a normed space X such that $X = Y'$), one may wonder whether in Theorem 4.5 the weak topology may be replaced by the weak star topology. If X is not reflexive, the answer is in the negative. For instance, for any $F \in Y'' \setminus Y$, the hyperplane $\{f \in Y' : F(f) = 0\}$ is not weakly star closed, [] although it is both strongly and weakly closed.

Corollary 4.6 (*Mazur*) *Let X be a normed space, and $u_n \to u$ weakly in X. Then u is the strong limit of a sequence $\{\tilde{u}_m\}$ of finite convex combinations of elements of the sequence $\{u_n\}$. That is, for any $m \in \mathbb{N}$ there exist $\ell_m \in \mathbb{N}$ and $\lambda_{m,k} \geq 0$ for $1 \leq k \leq \ell_m$ with $\sum_{k=1}^{\ell_m} \lambda_{m,k} = 1$ such that*

$$\tilde{u}_m := \sum_{k=1}^{\ell_m} \lambda_{m,k} u_k \ \to\ u \qquad \text{in } X, \text{ as } m \to \infty. \tag{4.7}$$

Proof. Let us set $S := \{u_n\}$, and notice that $\overline{\text{co}}(S)$ (the closure of the convex hull) is convex, as the closure of any convex set is convex. [Ex] By part (i) of Theorem 4.5, any weak limit of a sequence in S is included in $\overline{\text{co}}(S)$. Thus $u \in \overline{\text{co}}(S)$, that is, u is the (strong) limit of a sequence in $\text{co}(S)$. [24] □

*4.7 The theorem of Banach-Alaoglu

Let us recall that a subset A of a Hausdorff space H is called **relatively compact** (resp. **relatively sequentially compact**) **in** $\boldsymbol{H}$ iff its closure $\overline{A}$ is compact (resp. sequentially compact) when endowed with the topology induced by H. Thus, A is relatively sequentially compact in H iff every sequence in A has a convergent subsequence whose limit belongs to A.

A Hausdorff space H is said to be **(topologically) compact** iff a finite subcovering may be extracted from every open covering of that set. H is said to be **sequentially compact** iff a convergent subsequence may be extracted from every sequence. [25] In metrizable spaces the topological compactness is equivalent to the sequential compactness. However in nonmetrizable topological spaces in general there is no implication between topological and sequential compactness. For instance infinite-dimensional Banach spaces equipped with either the weak or (if they have a predual) weak star topology are nonmetrizable. [Ex]

Theorem 4.7 (*Banach-Alaoglu*) *Let X be a normed space. The closed unit ball of $B_{X'}$ is weakly star compact. (This entails that any bounded subset of X' is relatively weakly star compact.) []*

From this important result one may retrieve the theorem as it was originally stated by Banach: [26]

Theorem 4.8 (*Banach*) *Let X be a separable normed space. The closed unit ball of $B_{X'}$ is sequentially weakly star compact. (This entails that any bounded subset of X' is relatively sequentially weakly star compact.)*

See e.g. [Br] p. 34 for the proof. [27]

*4.8 Further results on weak topologies

The following holds in any infinite-dimensional normed space X.

[24] In passing note that in general $\overline{\text{co}}(A)$ (the closure of the convex hull) need not coincide with $\text{co}(\bar{A})$ (the convex hull of the closure). Consider e.g. $A = \{(x, x^2) : x \in \mathbb{R} \setminus \{0\}\}$.

[25] Since this definition does not refer to any ambient space, its limit must be an element of H.

[26] The hypotheses of Banach-Alaoglu are more general, but the thesis is weaker.

[27] There this result is erroneously attributed to Banach and Alaoglu. The corresponding footnote is also questionable ... (why?).

(i) As weakly open sets are unbounded, the open unit ball B_X^0 is not weakly open. It even has no interior point for the weak topology.

(ii) By Mazur's Theorem 4.5, the closed unit ball is weakly closed. This set coincides with its weak boundary, as it has no interior point in the weak topology.

(iii) The boundary of the unit ball, $\{v \in X : \|v\| = 1\}$, is closed but not weakly closed. For example, if $X = \ell^2$ the sequence of unit vectors $\{e_n\}$ weakly converges to the origin.

Here is a further property along the same line.

Proposition 4.9 *Let X be an infinite-dimensional normed space. The unit sphere $\{v \in X : \|v\| = 1\}$ in X is weakly dense in the closed unit ball B_X. The unit sphere $\{f \in X' : \|f\| = 1\}$ is also weakly star dense in $B_{X'}$.*

Proof. Let $u \in B_X$, and A be a weak neighborhood of u. As we saw, the weakly open set $A - u$ $(:= \{x - u : x \in A\})$ contains a straight line $\{tw : t \in \mathbb{R}\}$ for some $w \in X$, $w \neq 0$. Hence $\|u + tw\| = 1$ for a suitable $t \in \mathbb{K}$. Any weak neighborhood of any point of B_X thus contains a point of the unit sphere. We conclude that the unit sphere of X is weakly dense in B_X.

The proof of the second statement is quite similar. □

*4.9 Exercises

1. = Let A be a convex subset of a normed space X. Show that the closure of A coincides with its weak closure.

   (This obviously entails the first part of Mazur's Theorem 4.5.)

2. Let X be a separable infinite-dimensional normed space. Is the unit sphere $S_{X'}$ relatively sequentially weakly star compact?

3. Is a weakly closed set necessarily convex?

4. Let X be an infinite dimensional real normed space. Which set is the weak closure of the open unit ball B_X^0?

5. = Let X be a Banach space, B_X be the open unit ball of X, and set $A := B_X \setminus \{0\}$.
   (i) Is A relatively compact?
   (ii) What is the closure of A?
   (iii) What is the interior of A?
   (iv) What is the weak interior of A?
   (v) Is the weak closure of A bounded?

6. = Let $\{e_n\}$ be the canonical basis of ℓ^2, and set
   $A := \big\{ \sum_{n\in\mathbb{N}} a_n e_n : a_n \geq 0 \ \ \forall n, \ \ 1 \leq \sum_{n\in\mathbb{N}} |a_n|^2 \leq 2 \big\}$.
   (i) Is A closed?
   (ii) Is A compact?
   (iii) What is the interior of A?
   (iv) Is A sequentially weakly compact?
   (v) What is the weak interior of A?

7. For any of the spaces, $\mathbb{R}^4$, c, $L^p(\mathbb{R})$ $(1 \leq p \leq +\infty)$, establish whether

   (i) bounded subsets are sequentially compact,

   (ii) compact subsets are sequentially weakly compact,

   (iii) sequentially weakly compact subsets are compact,

   (iv) bounded subsets are relatively sequentially weakly star compact,

   * (v) closed bounded subsets are sequentially weakly star compact.

5 The Baire Theorem and its Consequences

*5.1 The Baire theorem

The results of this section stem from the following classical metric theorem.

Theorem 5.1 (*Baire Theorem) If a complete metric space X is a countable union of closed subsets, then at least one of them has interior points.*

* *Proof.* Let $\{X_n\}$ be any sequence of (possibly nondisjoint) closed subsets of X with empty interior. It suffices to show that $X \neq \bigcup_{n\in\mathbb{N}} X_n$.

As X_0 is closed and has no interior point, it cannot coincide with X. Thus $X \setminus X_0$ is open and contains a closed ball, say $B(x_0, \varepsilon_0)$. The set $X_1 \cap B(x_0, \varepsilon_0)$ is closed and has no interior point; $X \setminus [X_1 \cap B(x_0, \varepsilon_0)]$ is thus open and contains a closed ball $B(x_1, \varepsilon_1)$ with $\varepsilon_1 \leq \varepsilon_0/2$. Iterating this procedure we construct a nested sequence $\{B(x_n, \varepsilon_n)\}$ of closed balls; each of them does not intersect X_n, and $\varepsilon_n \to 0$. By the completeness of X, the sequence $\{x_n\}$ then converges to some $x \in X$. By construction $B(x_m, \varepsilon_m) \cap \big(\bigcup_{n\leq m} X_n\big) = \emptyset$ for any m. Hence $x \in \bigcap_{n\in\mathbb{N}} B(x_n, \varepsilon_n) \not\subset \bigcup_{n\in\mathbb{N}} X_n$; thus $X \neq \bigcup_{n\in\mathbb{N}} X_n$. □

The Baire theorem may be restated in a number of ways, e.g., the following formulation is simply obtained from Theorem 5.1 by taking complementary sets.
If X is a complete metric space, then:
the intersection of a countable family of open dense subsets of X is dense.

Notice that this theorem concerns (complete) metric spaces, but is stated in terms of topological notions.

5.2 The Principle of Uniform Boundedness

Let X_1 and X_2 be normed spaces with respective norms $\|\cdot\|_1$ and $\|\cdot\|_2$. [28] We know that the space $\mathcal{L}(X_1; X_2)$ of all linear and continuous operators $X_1 \to X_2$ is a normed space equipped with the operator norm

$$\|L\|_{\mathcal{L}(X_1;X_2)} = \sup\{\|Lu\|_2/\|u\|_1 : u \in X_1 \setminus \{0\}\} \; \big(= \sup\{\|Lu\|_2 : u \in X_1, \|u\|_1 \leq 1\}\big),$$

and is complete iff X_2 is complete. The principle of uniform boundedness states that any pointwise bounded family $\mathcal{F} \subset \mathcal{L}(X_1; X_2)$ is uniformly bounded.

Theorem 5.2 (*Banach-Steinhaus) Let X_1 be a Banach space, X_2 be a normed space, and $\mathcal{F} \subset \mathcal{L}(X_1; X_2)$. Then*

$$\sup_{L\in\mathcal{F}} \|Lu\|_2 < +\infty \quad \forall u \in X_1 \qquad \Rightarrow \qquad \sup_{L\in\mathcal{F}} \|L\|_{\mathcal{L}(X_1;X_2)} < +\infty. \tag{5.1}$$

This implication also reads as follows:

$$\begin{aligned} &\forall u \in X_1, \exists C > 0 : \forall L \in \mathcal{F}, \|Lu\|_2 \leq C\|u\|_1 \\ \Rightarrow \quad &\exists \hat{C} > 0 : \forall u \in X_1, \forall L \in \mathcal{F}, \|Lu\|_2 \leq \hat{C}\|u\|_1. \end{aligned} \tag{5.2}$$

Proof. For any $n \in \mathbb{N}$, let us set $A_n := \{u \in X_1 : \forall L \in \mathcal{F}, \|Lu\|_2 \leq n\}$. By the continuity of the operators of $\mathcal{F}$, this set is closed. By the assumption of pointwise boundedness, each $u \in X_1$

[28] Whenever we deal with operators between two normed spaces, we shall implicitly assume that they are linear spaces over the same field.

belongs to some A_n; that is, $\bigcup_n A_n = X_1$. Then, by Baire's theorem, for some $\widetilde{n} \in \mathbb{N}$ the interior of $A_{\widetilde{n}}$ is nonempty. So let $w \in X_1$ and $r > 0$ be such that the ball $B(w,r) := w + rB(0,1)$ is contained in $A_{\widetilde{n}}$. For any $L \in \mathcal{F}$ and any $v \in B(0,1)$, we then have $\|L(w+rv)\|_2 \le \widetilde{n}$, whence

$$r\|L(v)\|_2 \le \widetilde{n} + \|L(w)\|_2 \le \widetilde{n} + S(w), \quad S(w) := \sup_{L\in\mathcal{F}} \|Lw\|_2.$$

Hence $\|L\|_{\mathcal{L}(X_1;X_2)} \le [\widetilde{n} + S(w)]/r$ for all $L \in \mathcal{F}$. The implication (5.1) is thus established. □

The Banach-Steinhaus theorem fails if X_1 is not complete. As a counterexample, consider the family of functionals $\{L_n\}_{n\in\mathbb{N}}$ with $L_n x = n x_n$ for any $x := (x_1, x_2, ...) \in c_{00}$ and any n. This family of functionals are pointwise but not uniformly bounded. [Ex]

As another counterexample, let us denote by X_1 the linear space $C^1([0,1])$ equipped with the norm of $C^0([0,1])$, and consider the functionals $L_n u = n[u(1/n) - u(0)]$ for any $u \in X_1$ and any n.

In particular, the Banach-Steinhaus theorem result applies to functionals.

Corollary 5.3 *If X Banach space, then any pointwise bounded family of X' is uniformly bounded.*

Here are some other relevant consequencess of the Banach-Steinhaus theorem.

Corollary 5.4 *(i) If X is a normed space, then any weakly convergent sequence in X is bounded.*
(ii) If X is a normed space, then any weakly convergent sequence in X' is bounded.
(iii) If X is a Banach space, then any weakly star convergent sequence in X' is bounded. [Ex]

Concerning part (iii), notice that a weakly star convergent sequence $\{f_n\}$ in X' is tested over the elements of $J(X)$; that is, $\{f_n\}$ is regarded as a subset of $\mathcal{L}(J(X);\mathbb{R})$. The Banach-Steinhaus theorem $J(X)$ may then be applied only if $J(X)$ is complete; that is, as J is an isometric isomorphism, only if X is complete.

On the other hand, concerning part (ii), the weak convergence the sequence $\{f_n\}$ in X' is tested over the elements of X''; that is, $\{f_n\}$ is regarded as a subset of $\mathcal{L}(X'';\mathbb{R})$. As X'' is always complete, the Banach-Steinhaus theorem can be applied without assuming X to be complete. For a similar reason, in part (i) the completeness of X is not needed.

Corollary 5.5 *Let X_1 be a Banach space, X_2 be a normed space, and $\{L_n\}$ be a sequence in $\mathcal{L}(X_1;X_2)$. Assume that for any $u \in X_1$ the sequence $\{L_n u\}$ converges in X_2, and denote this limit by Lu. This defines an operator $L \in \mathcal{L}(X_1;X_2)$.*

(See [Br] p. 62 for the proof.)

5.3 The Open Mapping Theorem

A linear mapping between two normed spaces is called *open* iff it maps open sets onto open sets.

Lemma 5.6 *Let X_1 and X_2 be two normed spaces. A mapping $L \in \mathcal{L}(X_1;X_2)$ is open if (and only if) 0 is an interior point of $L(B_1)$. (Here B_1 denotes the open unit ball in X_1.)*

Proof. Let U be open in X. For any $u \in U$ there exists $\varepsilon > 0$ such that $u + \varepsilon B_1 \subset U$. If 0 is an interior point of $L(B_1)$, then $L(u)$ is an interior point of $L(u) + \varepsilon L(B_1) = L(u + \varepsilon B_1)\big(\subset L(U)\big)$, hence also of $L(U)$. □

This lemma entails that any open mapping is surjective. [Ex] The celebrated open mapping theorem states the converse, whenever X_1 and X_2 are complete.

Theorem 5.7 (Banach's Open Mapping Theorem) *Let X_1 and X_2 be Banach spaces. Any surjective mapping $L \in \mathcal{L}(X_1; X_2)$ is open.*

* *Proof.* Set $V_m = L(mB_1)$. We have $X_2 = \cup_{m\geq 1} V_m$ as L is surjective. By Baire's theorem, some $\overline{V_k}$ has nonempty interior, so it contains an open ball $v + B_\delta = v + \delta B_1$. We infer that $-v + B_\delta \subset \overline{V_k}$ by symmetry, and $B_\delta \subset \overline{V_k}$ by convexity; consequently $B_\varepsilon \subset \overline{V_1}$ for $\varepsilon = \delta/k$. Because of Lemma 5.6, it now suffices to prove that $\overline{V_1} \subset V_2$, since 0 will then also be an interior point of $V_1 = L(B_1)$. To this purpose, let $v \in \overline{V_1}$. Set $v_0 = v$ and choose $u_1 \in B_1$ with $v_1 := v_0 - Lu_1 \in B_{\varepsilon/2} \subset \overline{L(B_{1/2})}$. Proceeding inductively, choose $u_n \in B_{2^{1-n}}$ with $v_n := v_{n-1} - Lu_n \in B_{2^{-n}\varepsilon} \subset \overline{L(B_{2^{-n}})}$. Therefore $\sum_{n=0}^{\infty} \|u_i\| < \infty$; hence $\sum_{n=0}^{\infty} u_i$ converges in X_2. [29] We thus get

$$u := \sum_{n=1}^{\infty} u_n \in B_2, \quad v - \sum_{j=1}^{n} Lu_j = v_n \to 0;$$

thus $v = Lu \in L(B_2) = V_2$. □

Remarks. (i) One may wonder whether any surjective mapping $L \in \mathcal{L}(X_1; X_2)$ maps closed sets onto closed sets. This fails even in finite-dimensional spaces. For instance the linear, continuous and surjective functional $L : \mathbb{R}^2 \to \mathbb{R} : (x, y) \mapsto x$ maps the closed set $A := \{(x, y) : xy \geq 1\}$ onto the nonclosed set $L(A) = (0, +\infty)$.

(ii) Continuous mappings map compact sets to compact sets, without any linearity assumption. (These maps are not called *compact!)*

(iii) In the open mapping theorem the surjectivity cannot be dispensed with. The null function is a trivial counterexample.

Corollary 5.8 (*Inverse Mapping Theorem) Let X_1 and X_2 be Banach spaces. The inverse of any linear, continuous and bijective mapping $X_1 \to X_2$ is (linear and) continuous.*

Let us recall that, if $\|\cdot\|_1$ and $\|\cdot\|_2$ are two norms over a linear space, we say that the first is stronger than the second one iff there exists a constant $C > 0$ such that $\|u\|_2 \leq C\|u\|_1$ for all $u \in X$.

Corollary 5.9 *Let a linear space X be a Banach space when equipped with either of two norms. If one of them is stronger than the other one, then the two norms are equivalent.*

Proof. Denoting X_1 and X_2 the two normed spaces, it suffices to apply Corollary 5.8 to the identity mapping $X_1 \to X_2$. □

Exercise: two norms on the same Banach space are equivalent. (Hint: take the sum of the norms, and apply a corollary of the open mapping theorem.) But, surprisingly, this statement is wrong ! (it is not obvious that with the sum of the norms the space is complete...)

see Theorem 4.2 of Let E be an infinite dimensional Banach space. There exist mutually non-equivalent norms on E all making the space into an isometric copy of the original one. More precisely, the cardinality of such a set of norms can be chosen to be as large as the dimension

[29] In an exercise we proved that in a Banach space totally convergent sequences are convergent.

of E. in Arch. Math. 92 (2009), 414427 Equivalent complete norms and positivity Wolfgang Arendt and Robin Nittka

5.4 The Closed Graph Theorem

It is easy to see that, whenever X_1 and X_2 are two normed spaces (resp. Banach spaces) over the field $\mathbb{K}$, $X_1 \times X_2$ is a normed space (resp. a Banach space) over $\mathbb{K}$ when equipped with the norm

$$\|(v_1, v_2)\| := \|v_1\|_1 + \|v_2\|_2 \qquad \forall (v_1, v_2) \in X_1 \times X_2,$$

and that the projections $p_j : X_1 \times X_2 \to X_j$ are continuous. The next consequence of the Open Mapping Theorem is as simple as important.

Theorem 5.10 (*Closed Graph Theorem*) *Let X_1 and X_2 be Banach spaces, let $L : X_1 \to X_2$ be a linear mapping. The graph of L, $G_L := \{(v, Lv) : v \in X_1\}$, is closed in $X_1 \times X_2$ iff L is continuous.*

Proof. Let us first prove the "if"-part. For any$(u, w) \in \overline{G}_L$, there exists a sequence $\{u_n\}$ in X_1 such that $u_n \to u$ and $Lu_n \to w$. If L is continuous then $w = Lu$, that is, $(u, w) \in G_L$.

Let us next prove the "only if"-part. If the linear subspace G_L of $X_1 \times X_2$ is closed, then it is a Banach space by itself. The projections

$$p_i : G_L \to X_i, \qquad p_1(v, Lv) = v, \qquad p_2(v, Lv) = Lv,$$

are continuous. As p_1 is bijective, p_1^{-1} also is continuous by the inverse mapping theorem (Corollary 5.8). Therefore $L = p_2 \circ p_1^{-1}$ is continuous. □

Remarks. (i) Let $L : X_1 \to X_2$ be a linear mapping between two Banach spaces X_1 and X_2, and let us consider the following statements:

$$(a)\ u_n \to u \ \text{ in } X_1, \qquad (b)\ Lu_n \to w \ \text{ in } X_2, \qquad (c)\ w = Lu.$$

The closedness of L reads "(a) and (b) together imply (c)" whereas its continuity is tantamount to "(a) implies (b) and (c)" (for any sequence $\{u_n\}$ and any u in X_1). The fact that the closedness entails the continuity is a remarkable consequence of the open mapping theorem.

We emphasize that it is assumed that $L(X_1) = X_2$ in the open mapping theorem, and $\mathrm{Dom}(L) = X_1$ in the closed graph theorem.

(ii) Let X_1 and X_2 be Banach spaces, and $L : X_1 \to X_2$ be a linear mapping. The closedness of the graph of L (that is, the continuity of L) does not entail the closedness of the range of L. Whenever $X_1 \subset X_2$ with proper and continuous injection, the indentity mapping $X_1 \to X_2$ is a counterexample.

(iii) Each of the main three theorems of this section – the open mapping theorem (Theorem 5.7), the inverse mapping theorem (Corollary 5.8), the closed graph theorem (Theorem 5.10) – may be derived from any other one of them.

(iv) These theorems may be extended to Fréchet spaces (even to the more general environment of locally convex topological vector spaces!) with analogous proofs. Actually, they do not involve dual spaces and weak topologies.

*5.5 Exercises

1. * Let X, Y be Banach spaces and $T \in \mathcal{L}(X, Y)$. Is it true that T is surjective iff $T(B_x)$ is a neighbourhood of the origin?

2. Consider the following argument:

"For any $m \in \mathbb{N}$, let us define $f_m \in (c_{00})'$ by setting $f_m(u) = \sum_{n=0}^{m} u_n$ for any $u := (u_1, u_2, ...) \in c_{00}$. For any $u \in c_{00}$, the sequence $\{f_m(u)\}$ is clearly bounded (in $\mathbb{R}$). By the corollary of the Banach-Steinhaus theorem, the sequence $\{f_m\}$ is then bounded in $(c_{00})'$."

Show that the conclusion is wrong, and find the mistake in the argument.

3. = * Let X be an infinite-dimensional normed space. Construct a linear functional $f : X \to \mathbb{R}$ that is not continuous.

4. = ** Prove the following principle of uniform boundedness in metric spaces, adapting the argument of Theorem 5.2.

Theorem (Osgood Theorem). Let X be a complete metric space, N be a normed space, and $\mathcal{F}$ be a family of pointwise bounded, continuous functions $X \to N$. Then $\mathcal{F}$ is uniformly bounded on some nonempty open subset of X; that is,

$$\sup\{|f(x)| : f \in \mathcal{F}\} < +\infty \ \ \forall x \in X \quad \Rightarrow$$
$$\exists x_0 \in X, \exists r > 0 : \ \sup\{|f(x)| : f \in \mathcal{F}, d(x, x_0) < r\} < +\infty.$$

Hint: See the proof of the Banach-Steinhaus theorem...

5. Let U be a subset of a Banach space X. Show that

$$\sup\{|f(u)| : u \in U\} < +\infty \quad \forall f \in X', \|f\|_{X'} \leq 1$$
$$\Rightarrow \quad \sup\{|f(u)| : u \in U, f \in X', \|f\|_{X'} \leq 1\} < +\infty.$$

(This is a dual formulation of the Principle of Uniform Boundedness for functionals.)

6. Let X be a Banach space. Prove that:

(i) A subset A of X is bounded if (and only if) $\{f(v) : v \in A\}$ is bounded for any $f \in X'$.
(ii) A subset A of X' is bounded if (and only if) $\{F(f) : f \in A\}$ is bounded for any $F \in X''$.
(iii) A subset A of X' is bounded if (and only if) $\{f(v) : f \in A\}$ is bounded for any $v \in X$.

7. Let us denote by X the linear space $C^1([0,1])$ equipped with the norm of $C^0([0,1])$, and consider the following argument:

"The differentiation operator $L : X \to C^0([0,1]) : u \mapsto u'$ is linear and closed. By the Closed Graph Theorem, the operator L is then continuous."

Show that the conclusion is wrong, and find the mistake in the argument.

8. In the proof of Corollary 4.3 at p. 62 of [Br], where is the Banach-Steinhaus theorem applied?

9. Provide an alternative proof of the Inverse Mapping Theorem via the Closed Graph Theorem.

10. = * (i) Show that the dimension of a Banach space cannot be countable.
Hint: Use the Baire theorem...
* (ii) A normed space may have countable infinite dimension: give an example.

Remark. Although the completeness of a normed space is a metric property, the statement (ii) does not contradict the assertion that the dimension is a purely algebraic notion.

11. = ** Conjecture: the norms of two Banach spaces over the same set necessarily equivalent.

Tentative argument: The sum of the two norms is a norm over the same set. By the inverse mapping theorem, it is equivalent to the two given norms.

This argument is wrong: why?

(One may show that the conjecture is wrong, but this is nontrivial.)

6 Hilbert Spaces

6.1 Inner-Product spaces

Let H be a linear space over the field $\mathbb{K}$. A mapping $(\cdot,\cdot) : H \times H \to \mathbb{K}$ is called an **inner product** (or a **scalar product)** [30] over H iff it fulfills the following properties:

$$\text{the functional } H \to \mathbb{K} : u \mapsto (u,v) \text{ is linear over } \mathbb{K} \qquad \forall v \in H, \tag{6.1}$$

$$(u,v) = \overline{(v,u)} \qquad \forall u,v \in H, \tag{6.2}$$

the bar denoting complex conjugation[31], and

$$(u,u) > 0 \qquad \forall u \in H \setminus \{0\}\,. \tag{6.3}$$

In particular, the positive definiteness property (6.3) implies that $(u,u) \in \mathbb{R}$ for all $u \in H$ and that

$$(u,u) = 0 \quad \Leftrightarrow \quad u = 0\,. \tag{6.4}$$

The properties (6.1) and (6.2) obviously entail that

$$\text{the functional } H \to \mathbb{K} : v \mapsto \overline{(u,v)} \text{ is linear over } \mathbb{K} \quad \forall u \in H. \tag{6.5}$$

A linear space over $\mathbb{C}$ ($\mathbb{R}$, resp.) equipped with an inner product is called a complex (real, resp.) **inner-product space**, or a **pre-Hilbert space**. Here is some further terminology:
(6.1) and (6.5) $\Leftrightarrow$ $(\cdot,\cdot)$ is **sesquilinear** if $\mathbb{K} = \mathbb{C}$, **bilinear** if $\mathbb{K} = \mathbb{R}$.
(6.1) and (6.2) $\Leftrightarrow$ $(\cdot,\cdot)$ is **Hermitian** (or **skew-symmetric**) if $\mathbb{K} = \mathbb{C}$, **symmetric** if $\mathbb{K} = \mathbb{R}$.
(6.5) $\Leftrightarrow$ $(u,\cdot)$ is **antilinear**, or **conjugate-linear**, or **skew-linear**.

Henceforth, when dealing with an inner-product space, we set

$$\|u\| := \sqrt{(u,u)} \qquad \forall u \in H. \tag{6.6}$$

By the properties (6.1) – (6.3) above and by (6.9) ahead, we infer that $\|\cdot\|$ is indeed a norm over H; this is called a **Hilbert norm**. Dealing with aninner-product space we shall always refer to this norm, if not otherwise specified.

Remark. A Banach space equipped with a non-Hilbert norm may be **Hilbertizable**; that is, a non-Hilbert norm may be equivalent to a Hilbert norm. For instance, the linear space $L^2(0,1)$ equipped with the non-Hilbert norm $\|\cdot\|_{L^2} + \|\cdot\|_{L^1}$ is Hilbertizable, since this norm is equivalent to the Hilbert norm $\|\cdot\|_{L^2}$.

6.2 Basic properties

Proposition 6.1 *If H is an inner-product space over the field $\mathbb{K}$, then*

$$|(u,v)| \le \|u\|\|v\| \qquad \forall u,v \in H \quad \text{(Cauchy-Schwarz inequality)}, \tag{6.7}$$

$$\|u+v\|^2 + \|u-v\|^2 = 2\|u\|^2 + 2\|v\|^2 \quad \forall u,v \in H \text{ (parallelogram identity)}, \tag{6.8}$$

$$\|u+v\| \le \|u\| + \|v\| \qquad \forall u,v \in H \quad \text{(Minkowski inequality)}, \tag{6.9}$$

$$\textit{the mapping } (\cdot,\cdot) : H \times H \to \mathbb{K} \textit{ is continuous.} \tag{6.10}$$

[30] The notation $(\cdot,\cdot)$ is traditional. Unfortunately it is also used to denote pairs.
[31] Of course, for a real space the complex conjugation may be dropped.

Proof. (i) Let us prove (6.7). Without loss of generality, we may assume that $\mathbb{K} = \mathbb{C}$ and $v \neq 0$. By (6.1) and (6.2), [32]

$$\begin{aligned} 0 \leq (u + \lambda v, u + \lambda v) &= (u, u) + |\lambda|^2 (v, v) + (\lambda v, u) + (u, \lambda v) \\ &= \|u\|^2 + |\lambda|^2 \|v\|^2 + 2Re[\lambda(v, u)] \qquad \forall \lambda \in \mathbb{C}. \end{aligned}$$

By taking $\lambda = -(u, v)/\|v\|^2$, we then get

$$0 \leq \|u\|^2 + \frac{|(u, v)|^2}{\|v\|^2} - 2\frac{|(u, v)|^2}{\|v\|^2} = \|u\|^2 - \frac{|(u, v)|^2}{\|v\|^2},$$

which yields (6.7).

(ii) In order to check (6.8), notice that

$$\begin{aligned} \|u + v\|^2 = (u + v, u + v) &= \|u\|^2 + (u, v) + (v, u) + \|v\|^2 \\ &= \|u\|^2 + 2Re(u, v) + \|v\|^2, \end{aligned} \tag{6.11}$$

and similarly $\|u - v\|^2 = \|u\|^2 - 2Re(u, v) + \|v\|^2$. Summing these equalities we get (6.8).

(iii) By (6.11) and by the Cauchy-Schwarz inequality we have

$$\|u + v\|^2 \leq \|u\|^2 + 2\|u\|\|v\| + \|v\|^2 = \big(\|u\| + \|v\|\big)^2 \qquad \forall u, v \in H,$$

that is (6.9).

(iv) Let two sequences $\{u_n\}$ and $\{v_n\}$ in H be such that $u_n \to u$ and $v_n \to v$ in the norm topology. We have

$$\begin{aligned} |(u, v) - (u_n, v_n)| &\leq |(u, v) - (u_n, v)| + |(u_n, v) - (u_n, v_n)| \\ &\leq \|u - u_n\|\|v\| + \|u_n\|\|v - v_n\|; \end{aligned}$$

as $\|u_n\|$ is uniformly bounded, the latter sum vanishes as $n \to \infty$. □

It is promptly checked that the Cauchy-Schwarz inequality (6.7) is reduced to an equality iff u and v are linearly dependent. The same holds for the Minkowski inequality (6.9).

6.3 The polarization identity

The denomination of (6.8) as *parallelogram identity* is easily understood by considering the parallelogram of vertices $0, u, v, u + v$ in the two-dimensional subspace spanned by u and v (assuming $u, v \neq 0$ and $u \neq v$): by (6.8), the sum of the squared lengths of the sides equals the sum of the squared lengths of the diagonals. In the plane this is known as the Apollonius theorem.

We saw that the inner product determines a norm which fulfills the parallelogram identity (6.8). The polarization identity that is displayed in the next lemma relates the inner product to the corresponding norm, inverting the relation (6.6). More generally, Theorem 6.3 below shows that any normed space in which the norm fulfills the parallelogram identity (6.8) is an inner-product space, in which the inner product is defined via the polarization identity. The parallelogram identity thus characterizes inner-product spaces in the class of normed spaces, and characterizes Hilbert norms among the norms of a Hilbert space. [33] E.g., the norm of $C^0([0, 1])$ and that of ℓ^p with $p \neq 2$ do not fulfill the parallelogram identity.

[32] We shall still denote by $Re(z)$ the real part of any complex number z.

[33] The are several other characterizations. See e.g. [de Figueiredo-Karlovitz].

Lemma 6.2 *Let V be a complex linear space equipped with a sesquilinear mapping $b : V \times V \to \mathbb{C}$.* [34] *Let $q : V \to \mathbb{R}$ be the corresponding quadratic mapping, that is, $q(v) := b(v, v)$ for any $v \in V$. The following* polarization identity *then holds* [35]

$$\begin{aligned} b(u, v) &:= \frac{1}{4} \sum_{n=1}^{4} i^n q(u + i^n v) \\ &= \frac{1}{4} \big\{ [q(u + v) - q(u - v)] + i[q(u + iv) - q(u - iv)] \big\} \qquad \forall u, v \in H. \end{aligned} \tag{6.12}$$

If V is a real linear space and $b : V \times V \to \mathbb{R}$ is a bilinear mapping, then the i-terms drop; that is, (6.12) *is replaced by*

$$b(u, v) := \frac{1}{4}[q(u + v) - q(u - v)] \qquad \forall u, v \in V.[Ex] \tag{6.13}$$

Theorem 6.3 (P. Jordan, von Neumann) *If H is a complex normed space equipped with a norm $\| \cdot \|$ that fulfills the parallelogram identity* (6.8)*, then*

$$\begin{aligned} (u, v) &:= \frac{1}{4} \sum_{n=1}^{4} i^n \|u + i^n v\|^2 \\ &= \frac{1}{4} \big[(\|u + v\|^2 - \|u - v\|^2) + i(\|u + iv\|^2 - \|u - iv\|^2) \big] \qquad \forall u, v \in H \end{aligned} \tag{6.14}$$

defines an inner product, which is related to the norm $\| \cdot \|$ by (6.6)*.*

If H is a real normed space, the same holds with (6.14) *replaced by*

$$(u, v) := \frac{1}{4} \big(\|u + v\|^2 - \|u - v\|^2 \big) \qquad \forall u, v \in H.[Ex] \tag{6.15}$$

(In the real case one may also set $(u, v) := \frac{1}{2} \big(\|u + v\|^2 - \|u\|^2 - \|v\|^2 \big)$.)

6.4 Hilbert spaces

An inner-product space is called a **Hilbert space** whenever it is complete w.r.t. the induced norm. A (closed) Banach subspace of a Hilbert space is itself a Hilbert space, and is called a (closed) Hilbert subspace. Henceforth we shall confine ourselves to Hilbert spaces, although for several results the completeness is not needed.

Two Hilbert spaces H_1 and H_2 are called isometrically isomorphic iff there exists a linear surjective operator $U : H_1 \to H_2$ such that $(Uu, Uv)_{H_2} = (u, v)_{H_2}$ for any $u, v \in H_1$. Such an operator is called **unitary.**

Theorem 6.4 *Any Hilbert space is reflexive. []*

This result will also be retrieved as a consequence of the Riesz-Fréchet representation of the dual of Hilbert spaces, see Theorem 8.1 below.

6.5 Examples of Hilbert spaces

(i) For any $N \geq 1$, $\mathbb{K}^N$ is a Hilbert space over $\mathbb{K}$ equipped with the inner product

$$(u, v)_{\mathbb{K}^N} := \sum_{n=1}^{N} u_n \overline{v_n} \qquad \forall u, v \in \mathbb{K}^N. \tag{6.16}$$

[34] We are not assuming either b to be Hermitian or q to be positive definite.

[35] As usual, by i we denote the imaginary unit.

$\mathbb{C}^N$ can also be equipped with the structure of Hilbert spaces over $\mathbb{R}$: this corresponds to identifying $\mathbb{R}^{2N}$ with $\mathbb{C}^N$ via the mapping $(u_1, ..., u_{2N}) \mapsto (u_1 + iu_2, ..., u_{2N-1} + iu_{2N})$.

(ii) For any measure space $(A, \mathcal{A}, \mu)$, the space of $\mathbb{K}$-valued functions $L^2(A, \mathcal{A}, \mu)$ is a Hilbert space over $\mathbb{K}$ equipped with the inner product

$$(u, v) := \int_A u(x)\overline{v(x)}\, d\mu(x) \qquad \forall u, v \in L^2(A, \mathcal{A}, \mu). \tag{6.17}$$

(iii) As a particular case of the example (ii), the sequence space ℓ^2 $(= \ell^2_{\mathbb{K}})$ is a Hilbert space over $\mathbb{K}$ equipped with the inner product

$$(u, v) := \sum_{n=1}^{\infty} u_n \overline{v_n} \qquad \forall u = \{u_n\}, v = \{v_n\} \in \ell^2. \tag{6.18}$$

(iv) For any $N \geq 1$, one may also define

$$L^2(A, \mathcal{A}, \mu; \mathbb{K}^N) = L^2(A, \mathcal{A}, \mu; \mathbb{K})^N, \qquad L^2(A, \mathcal{A}, \mu; \mathbb{K}^{N\times N}) = L^2(A, \mathcal{A}, \mu; \mathbb{K})^{N\times N},$$ [36]

ℓ^N, $\ell^{N\times N}$ and so on, in a obvious way; these are also Hilbert spaces.

(v) The linear space of sequences of $\mathbb{K}$ that only contain a finite number of nonvanishing elements is a noncomplete inner-product subspace of ℓ^2. Its completion coincides with the Hilbert space ℓ^2.

(vi) Let Ω be a bounded open subset of $\mathbb{R}^N$ $(N \geq 1)$, Let us equip the linear space of continuous functions $\bar{\Omega} \to \mathbb{K}$ with the inner product

$$(u, v) := \int_\Omega u(x)\overline{v(x)}\, dx \qquad \forall u, v \in C^0(\bar{\Omega}).$$

This space is not complete. For instance, for $\Omega = \,]-1, 1[$, $\{u_n : x \mapsto \arctan(nx)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in this space, but it does not converge to any continuous function (it converges a.e. to the discontinuous function $\frac{\pi}{2}\text{sign}$). The completion of this space coincides with the Hilbert space $L^2(-1, 1)$. [Ex]

6.6 Exercises

1. Let V be a complex linear space and $b_1(\cdot, \cdot)$, $b_2(\cdot, \cdot)$ be two sesquilinear mappings $V \times V \to \mathbb{C}$ such that $b_1(v, v) = b_2(v, v)$ for any $v \in V$. Show that these two mappings then coincide on the whole $V \times V$.

2. Let V be a complex linear space and a mapping $b(\cdot, \cdot) : V \times V \to \mathbb{C}$ be either sesquilinear and symmetric (rather than skew-symmetric), or bilinear and skew-symmetric (rather than symmetric). Show that then in either case $b(u, v) = 0$ for any $u, v \in V$.

3. Let V be a complex linear space. Show that a sesquilinear mapping $(\cdot, \cdot) : V \times V \to \mathbb{C}$ is Hermitian iff the associated quadratic mapping $V \to \mathbb{C} : v \mapsto (v, v)$ is real-valued.

   Notice that this fails in real spaces (with "symmetric" in place of "Hermitian")!

4. Let H be the set of all complex sequences $\{x_n\}$ such that

$$\|\{x_n\}\| := \Big(\sum_{n=1}^{\infty} \|x_n\|^4\Big)^{1/4} + \Big(\sum_{n=1}^{\infty} \|x_n\|^2\Big)^{1/2} < +\infty.$$

   Check that this is a norm on H.

   (i) Is this norm equivalent to a Hilbert norm?
   (ii) Does H coincide with any of the known sequence spaces?
   (iii) Formulate an analogous exercise in terms of Lebesgue functions $]0, 1[\to \mathbb{R}$ (instead of sequences, with integrals instead of series), and answer the analogous questions.

[36] This is a space of matrix-valued functions.

5. Let H be an inner-product space and $x, y \in H$. Show that

$$x \perp y \quad \Leftrightarrow \quad \|x + \lambda y\| \geq \|x\| \quad \forall \lambda \in \mathbb{K}.$$

6. Let H be an inner-product space. Show that for any $x, y, z \in H$

$$\|x - z\| = \|x - y\| + \|y - z\| \quad \Leftrightarrow \quad \exists \lambda \in]0, 1[: \ y = \lambda x + (1 - \lambda) z.$$

7. Let H be an inner-product space and $x, y \in H$. Show that if H is a real space then

$$x \perp y \quad \Leftrightarrow \quad \|x + y\|^2 = \|x\|^2 + \|y\|^2,$$

and find a counterexample for a complex space.

8. Let H be a complex inner-product space and $x, y \in H$. Show that

$$x \perp y \quad \Leftrightarrow \quad \|ax + by\|^2 = a^2 \|x\|^2 + b^2 \|y\|^2 \quad \forall a, b \in \mathbb{C}.$$

9. Let $\{u_n\}$ be a sequence in a Hilbert space H and $u \in H$ be such that $u_n \to u$ weakly and $\|u_n\| \to \|u\|$. Prove that then $u_n \to u$.

   Hint: Develop the square $\|u_n - u\|^2$ ...

10. Let H be a Hilbert space, $L : \mathrm{Dom}(L) \subset H \to H$ be a closed unbounded linear operator, and equip the linear space $X = \{v \in H : Lv \in H\}$ with the norm $\|v\|_X = \|v\|_H + \|Lv\|_H$ [which is named *the norm of the graph* of L in H].

    (i) Check that this is a Banach space.
    (ii) Is this space Hilbertizable?

11. Let $1 \leq p \leq +\infty$. In the linear space ℓ^p set $|||\cdot||| = \|\cdot\|_{\ell^2} + \|\cdot\|_{\ell^p}$.

    (i) $|||\cdot|||$ is a norm for some p?
    (ii) Is the corresponding space complete? (if not so, indicate the associated completed space.)
    (iii) Is $|||\cdot|||$ a Hilbert norm for some p?
    (iv) Is $|||\cdot|||$ a Hilbertizable norm for some p?

12. * Exhibit a nonseparable Hilbert space.

    Hint: A standard example uses the Cartesian product of a continuous families of copies of the field $\mathbb{K}$...

13. Consider the following clases:

    $\mathcal{B}$: Banach spaces, $\mathcal{H}$: Hilbert spaces,

    $\mathcal{E}$: Euclidean spaces, $\mathcal{P}$: Normed spaces with a predual,

    $\mathcal{F}$: Fréchet spaces, $\mathcal{N}$: Normed spaces, $\mathcal{R}$: Reflexive spaces.

    Which inclusions hold among these spaces?

7 Orthogonality and Projections

The norm provides a distance that is homogeneous and invariant by translation. The inner product allows one to define angles, in particular orthogonality, and then orthogonal projections.

7.1 Orthogonality

Let H be an inner-product space. We shall say that two elements $u, v \in H$ are **orthogonal**, and write $u \perp v$, iff $(u, v) = 0$. More generally, we shall say that two (nonempty) subsets U and

V of H are orthogonal, and write $U \perp V$, iff $(u, v) = 0$ for any $u \in U$ and any $v \in V$. We define the **orthogonal complement** of any (nonempty) subset U of H as

$$U^{\perp} := \{v \in H : (v, u) = 0 \ \ \forall u \in U\}.$$

In real inner-product spaces one can measure angles: for any unit vectors $u, v \in H$ we define the (nonoriented) angle formed by u and v to be $\arccos(u, v)$. There is no analogous notion in general Banach spaces.

Theorem 7.1 (*Orthogonal Projection on a Convex Set) Let K be a nonempty closed convex subset of a Hilbert space H. For any $u \in H$ there exists one and only one* ***(orthogonal) projection*** *$w \in K$ such that*

$$\|u - w\| = \inf\{\|u - v\| : v \in K\}. \tag{7.1}$$

This condition is equivalent to the ***variational inequality***

$$Re\,(u - w, v - w) \le 0 \qquad \forall v \in K. \tag{7.2}$$

The projection operator $P_K : u \mapsto w$ is ***nonexpansive****, that is,*

$$\|P_K u_1 - P_K u_2\| \le \|u_1 - u_2\| \qquad \forall u_1, u_2 \in H. \tag{7.3}$$

A geometric interpretation provides a clear understanding of this theorem. For instance, by drawing the intersection of K with the plane that contains u, w and v (assuming that they are distinct and nonaligned), explains why (7.2) characterizes the orthogonal projection. A similar geometric reduction may also be used for some other results of the theory of Hilbert spaces.

Proof. (i) Let $\{v_n\} \subset K$ be a **minimizing sequence** for the distance from K, that is,

$$d_n := \|u - v_n\| \to \inf\{\|u - v\| : v \in K\} =: d \qquad \text{as } n \to \infty.$$

The parallelogram identity yields

$$\begin{aligned} &2\|u - (v_n + v_m)/2\|^2 + 2\|(v_n - v_m)/2\|^2 \\ &= \|u - v_n\|^2 + \|u - v_m\|^2 = d_n^2 + d_m^2. \end{aligned}$$

As $(v_n + v_m)/2 \in K$, we have $\|u - (v_n + v_m)/2\| \ge d$. The preceding equality then yields

$$2\|(v_n - v_m)/2\|^2 \le d_n^2 + d_m^2 - 2d^2 \to 0.$$

Thus $\{v_n\}$ is a Cauchy sequence in H; by the completeness of H, it converges to some $w \in K$. The continuity of the norm then yields (2.1).

(ii) Let w fulfil (7.1). For any $v \in K$ and any $t \in]0, 1]$, $w + t(v - w) \in K$ by the convexity of K. Hence

$$\begin{aligned} \|u - w\|^2 &\le \|u - [w + t(v - w)]\|^2 \\ &= \|u - w\|^2 - 2tRe(u - w, v - w) + t^2\|v - w\|^2, \end{aligned}$$

that is, $2t\,Re\,(u - w, v - w) \le t^2\|v - w\|^2$. Dividing by t and passing to the limit as $t \to 0$, we then get (7.2). Conversely,

$$\begin{aligned} \|u - v\|^2 &= \|(u - w) - (v - w)\|^2 \\ &= \|u - w\|^2 + \|v - w\|^2 - 2Re(u - w, v - w) \\ &\overset{(7.2)}{\ge} \|u - w\|^2 + \|v - w\|^2 \ge \|u - w\|^2 \qquad \forall v \in K. \end{aligned}$$

(iii) For any given $u_1, u_2 \in H$, let $w_1, w_2 \in H$ satisfy (7.2) with $u = u_1$ resp. $u = u_2$, and take $v = w_2$ resp. $v = w_1$. Summing the two inequalities we get

$$\|w_1 - w_2\|^2 \leq Re\,(u_1 - u_2, w_1 - w_2) \leq \|u_1 - u_2\|\,\|w_1 - w_2\|,$$

whence $\|w_1 - w_2\| \leq \|u_1 - u_2\|$, i.e. (7.3). In particular, the choice $u_1 = u_2$ shows that (7.2) defines a unique element $w = P_K u$ for any given $u \in H$. □

Corollary 7.2 (*Orthogonal Projection on a Subspace) Let V be a closed subspace of a Hilbert space H. The projection operator P_V is then linear and continuous. Moreover, for any $u \in H$,*

$$w = P_V u \qquad \Leftrightarrow \qquad (w - u, v) = 0 \quad \forall v \in V. \tag{7.4}$$

The latter is called a **variational equation**, and the linear and continuous operator P_V is named an **(orthogonal) projection.**

Proof. Let us assume that $w = P_V u$. For any $\tilde{v} \in V$ and any real $\lambda > 0$, let us take $v = w \pm \lambda\tilde{v}$ ($\in V$) in (7.2), and then divide by λ. This yields $Re(w - u, \tilde{v}) = 0$ for any $\tilde{v} \in V$. By taking $\tilde{v} = v$ and $\tilde{v} = iv$ for any $v \in V$, we get $(w - u, v) = 0$. The converse implication and the linearity of P_V are straightforward. The continuity follows from (7.3). □

Remarks. (i) In Theorem 7.1 the distance from the nonempty closed convex set K is minimized without assuming any compactness property for K.

(ii) Theorem 7.1 rests upon the completeness of the set K, rather than H. Therefore this result remains valid in uncomplete inner-product spaces, provided that the convex subset K is complete. In particular the orthogonal projection either on a finite-dimensional linear subspace V, or on a (nonempty) closed convex subset of V, thus exists also in uncomplete inner-product spaces.

(iii) Variational inequalities and variational equations are extensively used in analysis, in particular in convex analysis, in optimization, in the analysis of PDEs, and so on. They also found a large number of applications in mathematical physics, in optimization, and so on.

Theorem 7.3 (*Orthogonal Decomposition) Let M be a linear subspace of a Hilbert space H. Then $M^\perp$ is a closed subspace, and* [37]

$$\bar{M} = \mathcal{R}(P_{\bar{M}}), \qquad M^\perp = \mathcal{N}(P_{\bar{M}}), \qquad H = \bar{M} \oplus M^\perp. \tag{7.5}$$

Moreover $M^\perp = (\bar{M})^\perp$ and $(M^\perp)^\perp = \bar{M}$.

Proof. For any sequence $\{u_n\}$ in H, if $u_n \perp M$ for any n and $u_n \to u$, then $u \perp M$, by the continuity of the inner product; thus $M^\perp$ is closed and $M^\perp \subset (\bar{M})^\perp$. The opposite inclusion is trivial.

For any $u \in H$, $u - P_{\bar{M}}u \in M^\perp$ by (7.4); thus $u = P_{\bar{M}}u + (u - P_{\bar{M}}u) \in \bar{M} + M^\perp$. Moreover, if $u \in \bar{M} \cap M^\perp$ then $(u, u) = 0$, that is, $u = 0$. The third equality in (7.5) is thus established.

Applying (7.5) to $M^\perp$ we get $H = M^\perp \oplus (M^\perp)^\perp$. Comparing this equality with (7.5), we conclude that $(M^\perp)^\perp = \bar{M}$. □

* **Remark.** By the formula $H = \bar{M} \oplus M^\perp$, any closed subspace of a Hilbert space is topologically complemented. By a celebrated theorem of Lindenstrauss and Tzafriri, the existence of a topological complement characterizes Hilbert spaces within the class of Banach spaces. []

[37] As usual, for any linear mapping L, we denote its range by $\mathcal{R}(L)$ and its nullspace by $\mathcal{N}(L)$. For any linear subspaces A and B, by "$H = A \oplus B$" we mean that $H = A + B$ and $A \cap B = \{0\}$.

Theorem 7.4 (*Characterization of Orthogonal Projections) For any closed subspace M of a Hilbert space H, the projection operator P_M is continuous, and*
(i) P_M is ***idempotent****, i.e. $P_M^2 = P_M$;*
(ii) P_M is ***self-adjoint****, i.e. $(P_M u, v) = (u, P_M v)$ for any $u, v \in H$.*
Conversely, any idempotent, self-adjoint, linear operator $P : H \to H$ coincides with the projection on the closed subspace $\mathcal{R}(P)$.

Proof. The continuity directly follows from the nonexpansiveness (7.3).
For any $u \in H$, $P_M u = P_M u + 0 \in M + M^\perp$, whence $P_M(P_M u) = P_M u$. Thus (i) holds.
For any $u, v \in H$, (7.4) yields $(P_M u, v) = (P_M u, P_M v) = (u, P_M v)$, i.e. (ii).
Let us now assume that $P : H \to H$ is an idempotent, self-adjoint, linear operator, and set $M := P(H)$; it will suffice to show that $P = P_M$. Properties (i) and (ii) and the Cauchy-Schwarz inequality yield

$$\|Pu\|^2 = (Pu, Pu) = (P^2 u, u) = (Pu, u) \le \|Pu\| \, \|u\|,$$

whence $\|Pu\| \le \|u\|$; thus P is continuous.
M is a linear subspace, and $Pv = v$ for any $v \in M$. For any sequence $\{u_n\}$ in M, if $u_n \to u$ then $Pu = \lim Pu_n = \lim u_n = u$. Thus M is a closed subspace of H. For any $u, v \in H$, as P is self-adjoint we have

$$(u - Pu, Pv) = (Pu - P^2 u, v) = (Pu - Pu, v) = 0.$$

Hence $u - Pu \in M^\perp$; thus $P = P_M$. □

Here is another characterization.

Theorem 7.5 *An operator $P \in \mathcal{L}(H)$ is a projection iff $P^2 = P$ and $\|P\|_{\mathcal{L}(X)} \le 1$. []*

7.2 Overview of Projections

So far we have seen three notions of linear projection:
(i) In linear spaces any *linear idempotent* operator from the space to itself is called a projection. The range is a linear subspace.
(ii) In Banach spaces one introduces *continuous* projections (or just projections). Here the range is a (closed) subspace.
(iii) In Hilbert spaces self-adjoint continuous projections are called *orthogonal* projections (or just projections).

In Hilbert spaces one may also deal with orthogonal projections on (nonempty) closed convex subsets: they map any element of the space to the point of the subset that has minimal distance. These projections can be expressed via variational inequalities, whereas projections on closed subspaces are characterized by variational equations.

For an orthogonal projection P in a Hilbert space we have $\|P\|_{\mathcal{L}(H)} \le 1$. Equality holds iff $P(H) \ne \{0\}$. On the other hand for a projection P in a Banach space X we may have $\|P\|_{\mathcal{L}(X)} > 1$, as trivial examples in $\mathbb{R}^2$ show.

7.3 Exercises

1. ***** Let H be a Hilbert space and M, N be two (closed) subspaces. Show that then:
   (i) The composition $P_M P_N$ is an orthogonal projection iff P_M and P_N commute, i.e. $P_M P_N = P_N P_M$. Either property entails $P_M P_N = P_{M \cap N}$.
   (ii) The sum $P_M + P_N$ is an orthogonal projection iff P_M and P_N reciprocally annihilate, i.e. $P_M P_N = P_N P_M = 0$, or equivalently $P_M \perp P_N$. Any of these properties entails $P_M + P_N = P_{M \oplus N}$.
   (iii) $M \subset N$ iff either $P_M P_N = P_N$, or $P_N P_M = P_N$, or $\|P_M x\| \le \|P_N x\|$ for any $x \in H$.

8 The Representation Theorem

8.1 The Riesz-Fréchet Theorem

Next from orthogonality we infer one of the key properties of Hilbert spaces.

Theorem 8.1 (*Riesz-Fréchet's Representation Theorem) Let H be a Hilbert space over the field $\mathbb{K}$. The operator*

$$\mathcal{R} : H \to H' \qquad \textit{defined by} \qquad \mathcal{R}_v(u) := (u, v) \qquad \forall u, v \in H \tag{8.1}$$

is bijective and isometric. H' is thus a Hilbert space, and is isometrically isomorphic to H.
If $\mathbb{K} = \mathbb{R}$ the mapping $v \to \mathcal{R}_v$ is linear, whereas if $\mathbb{K} = \mathbb{C}$ it is ***antilinear,*** *i.e.,*

$$\mathcal{R}_{\lambda_1 v_1 + \lambda_2 v_2} = \bar{\lambda}_1 \mathcal{R}_{v_1} + \bar{\lambda}_2 \mathcal{R}_{v_2} \qquad \forall \lambda_1, \lambda_2 \in \mathbb{C}, \forall v_1, v_2 \in H. \tag{8.2}$$

The inverse operator $R^{-1} : H' \to H$ is often called the **Riesz-Fréchet isomorphism**.

Proof. For any $u \in H$, by the Cauchy-Schwarz inequality (6.7) we have

$$|\mathcal{R}_v(u)| = |(u, v)| \le \|u\| \|v\| \,,$$

thus $\mathcal{R}_v \in H'$ and $\|\mathcal{R}_v\|_{H'} \le \|v\|$. As $\|v\|^2 = \mathcal{R}_v(v) \le \|\mathcal{R}_v\|_{H'} \|v\|$, the opposite inequality is also fulfilled. Thus $\mathcal{R}$ is an isometry.

Let us now fix any $f \in H'$ and show that $f = \mathcal{R}_v$ for some $v \in H$. Obviously, $\mathcal{R}_0 = 0$. Let us thus assume that $f \ne 0$, and choose any $z \in [f^{-1}(0)]^\perp$ with $f(z) = 1$. For any $u \in H$, $w := u - f(u)z \in f^{-1}(0)$. Hence $(w, z) = 0$, i.e. $(u, z) - f(u)\|z\|^2 = 0$. Setting $v = \|z\|^{-2} z$ we then get $(u, v) = f(u)$. Thus $R_v = f$ and therefore R_v is onto H'.

The final statement is a straightforward consequence of the antilinearity of the inner product w.r.t. to the second argument. □

Remarks. (i) This representation theorem generalizes Theorem 1.7 ($(L^2)' = L^2$, freely speaking), due to the same authors.

(ii) For any $u \in H$ one may also consider the functional $\tilde{\mathcal{R}}_u : H \to \mathbb{K} : v \mapsto (u, v)$. If $\mathbb{K} = \mathbb{R}$, $\tilde{\mathcal{R}}_u = \mathcal{R}_u$ and is linear. On the other hand if $\mathbb{K} = \mathbb{C}$, for any $u \in H$, $\tilde{\mathcal{R}}_u$ is continuous and antilinear:

$$\tilde{\mathcal{R}}_u(\lambda_1 v_1 + \lambda_2 v_2) = \bar{\lambda}_1 \tilde{\mathcal{R}}_u(v_1) + \bar{\lambda}_2 \tilde{\mathcal{R}}_u(v_2) \qquad \forall \lambda_1, \lambda_2 \in \mathbb{C}, \forall u, v_1, v_2 \in H.$$

Thus the map $u \mapsto \tilde{\mathcal{R}}_u$ is linear, but it maps H to its **antidual** $\tilde{H}$, namely the linear space of continuous and antilinear functionals $H \to \mathbb{C}$. [38] Notice that the operator

$$T : \tilde{H} \to H', \qquad T_{\tilde{f}}(v) = \overline{\tilde{f}(v)} \qquad \forall v \in H, \forall \tilde{f} \in \tilde{H}$$

is bijective, isometric, and linear. □

Proposition 8.2 *Any Hilbert space H is reflexive.*

Proof. We have to show that the canonical embedding $J : H \to H''$ is surjective. Let us define the antilinear operator $\mathcal{R} : H \to H'$ as in (8.1). For any $u'' \in H''$, the functional

[38] The occurrence of antilinearity is intrinsic, and there is no way to get rid of it.

$v \mapsto f(v) := \overline{\langle u'', \mathcal{R}_v \rangle}$ is an element of H'. By Theorem 8.1, then $f = \mathcal{R}_u$ for some $u \in H$. Denoting by J the canonical isomorphism $H \to H''$, we then have

$$\langle u'', \mathcal{R}_v \rangle = \overline{f(v)} = \overline{\mathcal{R}_u(v)} = \overline{(v,u)} = (u,v) = \mathcal{R}_v(u) = \langle J(u), \mathcal{R}_v \rangle \qquad \forall v \in H.$$

As $\mathcal{R}$ is surjective, $u'' = J(u)$. As u'' was arbitrary, we conclude that J is surjective. □

8.2 The Lax-Milgram Theorem

Theorem 8.3 (*Lax-Milgram*) *Let H be a Hilbert space, and $L \in \mathcal{L}(H)$ be such that, for some* $\alpha > 0$,

$$(Lv, v) \geq \alpha \|v\|^2 \qquad \forall v \in H \quad \text{(coerciveness)}. \tag{8.3}$$

Then L is bijective, and

$$\|L^{-1}w\| \leq \alpha^{-1}\|w\| \qquad \forall w \in H. \tag{8.4}$$

(Thus $L^{-1} \in \mathcal{L}(H)$ and $\|L^{-1}\|_{\mathcal{L}(H)} \leq \alpha^{-1}$.)

Proof. By the continuity and the coerciveness of L,

$$\alpha\|v\|^2 \leq (Lv, v) \leq \|Lv\|\,\|v\| \quad \text{whence} \quad \alpha\|v\| \leq \|Lv\| \qquad \forall v \in H;$$

whenever the latter inequality is fulfilled, one says that the operator L is **bounded below.** This entails that:

(i) L is injective;

(ii) if L^{-1} exists then (8.4) is fulfilled;

(iii) Any sequence $\{v_n\}$ in H is Cauchy if so is $\{Lv_n\}$.

To conclude the proof it then suffices to show that L is surjective. By (iii) and by the continuity of L, $L(H)$ is closed. For any $v \in L(H)^\perp$ we have $\alpha\|v\|^2 \leq (Lv, v) = 0$, whence $v = 0$; thus $L(H)^\perp = \{0\}$. As by Theorem 7.5 $H = L(H) \oplus L(H)^\perp = L(H) \oplus \{0\}$, we conclude that $L(H) = H$. □

Remark. This theorem generalizes the Riesz-Fréchet representation Theorem 8.1 to nonsymmetric operators. We check this assuming that H is a *real* Hilbert space, for the sake of simplicity. If $(Au, v) = (Av, u)$ for any $u, v \in H$, then $(u, v) \mapsto ((u, v)) := (Au, v)$ is a scalar product over H; moreover, by the continuity and coerciveness of A, the corresponding norm is equivalent to the original one. Then, by the representation theorem, for any $f \in H'$ setting $u_f := R^{-1}f \in H$ we have $((u_f, v)) = \langle f, v \rangle$ for any $v \in H$, i.e., $Au_f = f$.

The Lax-Milgram theorem is a widely used tool to derive existence and uniqueness results for linear boundary value problems, written as an operator equation $Au = b$ between suitably chosen function spaces.

8.3 Exercises

1. = Prove the following extension of the classical Pythagoras's theorem to Hilbert spaces. For any finite orthogonal system $\{u_n\}_{n=1,\ldots,m}$ of an inner-product space H,

$$\left\| \sum_{n=1}^{m} u_n \right\|^2 = \sum_{n=1}^{m} \|u_n\|^2.$$

For $m = 2$ (and only in this case), conversely this formula holds only if u_1 and u_2 are orthogonal.

2. Prove directly the Hahn-Banach Theorem I.4.1 in a Hilbert space, assuming that M is a closed subspace, without using the Zorn Lemma (neither any equivalent statement).

3. Let M be a closed subspace of a Hilbert space H, X be a Banach space, and $L : M \to X$ be a linear and continuous operator. Prove that L has a linear and continuous extension to the whole H.

   This may be regarded as a sort of Hahn-Banach-type Theorem for operators. Notice that this property may fail if H is just a Banach space. Consider for instance $M = X = c_0$, $H = \ell^\infty$ and L equal to the identity operator.

4. Let M be a linear subspace of a Hilbert space H. Prove that M is dense in H iff $M^\perp = \{0\}$.

5. Under which assumptions is $M^\perp = [(M^\perp)^\perp]^\perp$ in a Hilbert space?

6. Let H be the linear space of functions $f : \mathbb{R} \to \mathbb{R}$ such that $\{x \in \mathbb{R} : f(x) \neq 0\}$ is at most countable. Is this a Hilbert space w.r.t. the inner product $(f, g) := \sum_{x \in \mathbb{R}} f(x)g(x)$?

7. * Prove that, if H is a Hilbert space and $L \in \mathcal{L}(H)$ is symmetric, that is,

$$(Lu, v) = \overline{(u, Lv)} \ \big(= (Lv, u)\big) \qquad \forall u, v \in H,$$

   then the thesis of the Lax-Milgram theorem follows from the Riesz-Fréchet representation Theorem 8.1.

   Hint: The mapping $(u, v) \mapsto ((u, v)) := (u, Lv)$ defines an inner product over H. Notice that the dual H' is the same for the original and this newly defined inner product.

   Since L is continuous and coercive, the corresponding norm is equivalent to the original one, hence the dual H' is the same for either choice of inner product.

   To prove that L is surjective, let $b \in H$ define $f \in H'$ be setting $f(v) = (v, b)$ for any v. By the representation theorem, there exists $u \in H$ with $f(v) = ((v, u)) = (v, Lu)$ for any $v \in H$; thus $Lu = b$ and consequently L is surjective. That L is injective, and that (8.4) holds, follows as above by virtue of the inequality $\alpha\|v\| \le \|Lv\|$, valid for any $v \in H$.

9 Orthonormal Systems and Hilbert Bases

9.1 Orthonormal Systems

An either finite or countable or uncountable subset $A \neq \emptyset$ of an inner-product space H is called an **orthogonal system** iff $(u, v) = 0$ for any two distinct elements $u, v \in A$ (the case $0 \in A$ is not excluded). A is said to be **orthonormal** iff moreover any $u \in A$ is *normalized,* i.e., $\|u\| = 1$. (The origin 0 may thus belong to orthogonal but not to orthonormal systems).

Proposition 9.1 (*Gram-Schmidt Orthonormalization) Let $\{u_n\}$ be an either finite or countable linearly independent subset of an inner-product space H. Let us set $v_1 = u_1/\|u_1\|$. For any integer $n > 1$, by induction let us assume that $v_1, ..., v_n$ are known, and set*

$$\begin{aligned} w_{n+1} &:= u_{n+1} - \sum_{j=1}^{n} (u_{n+1}, v_j)v_j, \\ v_{n+1} &:= w_{n+1}/\|w_{n+1}\| \qquad \forall n \ge 1. \end{aligned} \tag{9.1}$$

This entails that $\{v_n\}$ is an orthonormal subset of H, and

$$span\{v_1, ..., v_n\} = span\{u_1, ..., u_n\} \qquad \forall n.$$

Proof. For any $n \in \mathbb{N}$, let us denote by V_n the span of $\{u_1, ..., u_n\}$. By induction hypothesis, let us assume that V_n coincides with the span of $\{v_1, ..., v_n\}$. Notice that $w_{n+1} \neq 0$ as $u_{n+1} \notin V_n$. For any n, by construction v_{n+1} is of unit norm and is orthogonal to $v_1, ..., v_n$. The sequence $\{u_n\}$ is thus orthonormal, and V_{n+1} coincides with the span of $\{v_1, ..., v_{n+1}\}$. □

In passing, notice that $\sum_{j=1}^{n}(u_{n+1}, v_j)v_j$ coincides with the projection of u_{n+1} on the span V_n of $\{v_1, ..., v_n\}$; this projection is well defined, as V_n is finite-dimensional.

For instance, the Gram-Schmidt procedure transforms the set of monomials $\{f_n(x) := x^n : n \in \mathbb{N} \cup \{0\}\}$ to an orthonormal subset of $L^2(0,1)$, more specifically the family of the classical Legendre polynomials. []

The next result illustrates the relevance of orthogonal sequences in Hilbert spaces.

Theorem 9.2 *For any orthogonal sequence $\{u_n\}_{n\in\mathbb{N}}$ in a Hilbert space H, the following properties are mutually equivalent:*

$$\sum_{n=1}^{\infty} u_n \quad \textit{converges unconditionally (in } H), \tag{9.2}$$

$$\sum_{n=1}^{\infty} u_n \quad \textit{converges weakly unconditionally (in } H), \tag{9.3}$$

$$\sum_{n=1}^{\infty} \|u_n\|^2 \quad \textit{converges (in } \mathbb{R}). \tag{9.4}$$

Proof. (9.2) ⇒ (9.3): this is obvious.

Let us show that (9.3) ⇒ (9.4). By (9.3) the sequence of the partial sums $\{\sum_{n=1}^{m} u_n\}_{m\in\mathbb{N}}$ is bounded. By the orthogonality of the sequence $\{u_n\}$, then

$$\sum_{n=1}^{m} \|u_n\|^2 = \left\| \sum_{n=1}^{m} u_n \right\|^2 \leq \text{Constant (independent of } m);$$

(9.4) then follows.

Let us next prove that (9.4) ⇒ (9.2). By (9.4) the sequence of partial sums $\{\sum_{n=1}^{m} \|u_n\|^2\}_{m\in\mathbb{N}}$ is Cauchy in $\mathbb{R}$, and by the orthogonality of $\{u_n\}$

$$\left\| \sum_{n=m}^{\ell} u_n \right\|^2 = \sum_{n=m}^{\ell} \|u_n\|^2 \qquad \forall \ell, m \in \mathbb{N}, m < \ell.$$

Hence the sequence of partial sums $\{\sum_{n=1}^{m} u_n\}_{m\in\mathbb{N}}$ is Cauchy in H. By the completeness of H, (9.2) then follows.

The convergence in (9.2) is obviously *unconditional* (i.e., it is invariant by reordering of the sequence — see below). The convergences in (9.3) and (9.4) are then also unconditional. □

By the latter Theorem, in a Hilbert space a series of orthogonal elements thus converges iff the sequence of the norms is an element of ℓ^2. Notice that in a Banach space a series converges if the sequence of the norms is an element of ℓ^1 ($\subset \ell^2$!).

9.2 Bessel inequality

For any (nonempty) subset A of a normed space X, we denote by $\overline{\text{span}}\,(A)$ is the closure of the set of finite linear combinations of elements of A; this is a closed subspace of H. Moreover, for any Hilbert space H,

$$\overline{\text{span}}\,(A) = (A^{\perp})^{\perp} \qquad \forall A \subset H.[Ex] \tag{9.5}$$

Lemma 9.3 *Let $\{u_n\}_{1\le n\le m}$ be a finite orthonormal family in an inner-product space H. Then for any $u \in H$*

$$\left\| u - \sum_{n=1}^{m} (u, u_n) u_n \right\|^2 = \|u\|^2 - \sum_{n=1}^{m} |(u, u_n)|^2 = \|u\|^2 - \left\| \sum_{n=1}^{m} (u, u_n) u_n \right\|^2. \tag{9.6}$$

Moreover $\sum_{n=1}^{m}(u, u_n)u_n$ coincides with the orthogonal projection of u on the closed subspace $\overline{span}\,(\{u_n\}_{1\le n\le m})$.

* *Proof.* Let us fix any $u \in H$, and set $\alpha_n := (u, u_n)$ for any $n \in \mathbb{N}$. By the orthonormality of $\{u_n\}$, for any $m \in \mathbb{N}$ we have

$$\left\| \sum_{n=1}^{m} \alpha_n u_n \right\|^2 = \sum_{n=1}^{m} \|\alpha_n u_n\|^2 = \sum_{n=1}^{m} |\alpha_n|^2 \qquad \forall \ell > m, \tag{9.7}$$

$$Re\left(u, \sum_{n=1}^{m} \alpha_n u_n \right) = \sum_{n=1}^{m} \|\alpha_n u_n\|^2 = \sum_{n=1}^{m} |\alpha_n|^2. \tag{9.8}$$

Hence

$$\begin{aligned} \left\| u - \sum_{n=1}^{m} \alpha_n u_n \right\|^2 &= \|u\|^2 - 2Re\left(u, \sum_{n=1}^{m} \alpha_n u_n \right) + \left\| \sum_{n=1}^{m} \alpha_n u_n \right\|^2 \\ &\overset{(9.7),(9.8)}{=} \|u\|^2 - 2\sum_{n=1}^{m} |\alpha_n|^2 + \sum_{n=1}^{m} |\alpha_n|^2 \\ &= \|u\|^2 - \sum_{n=1}^{m} |\alpha_n|^2 \qquad \forall m \in \mathbb{N}. \end{aligned} \tag{9.9}$$

For any m, let us define the partial sum $s_m = \sum_{n=1}^{m} \alpha_n u_n$, and notice that

$$\|s_\ell - s_m\|^2 = \left\| \sum_{n=m+1}^{\ell} \alpha_n u_n \right\|^2 = \sum_{n=m+1}^{\ell} |\alpha_n|^2 \qquad \forall \ell, m \in \mathbb{N}, m < \ell; \tag{9.10}$$

by the completeness of H, $\tilde{u} := \sum_{n=1}^{\infty} \alpha_n u_n$ then converges.
(9.7) and (9.9) also yield

$$\left\| u - \sum_{n=1}^{m} \alpha_n u_n \right\|^2 = \|u\|^2 - \left\| \sum_{n=1}^{m} \alpha_n u_n \right\|^2 \qquad \forall m \in \mathbb{N}\,. \tag{9.11}$$

A simple calculation shows that $(u - \tilde{u}, \tilde{u}) = 0$, and this yields the final statement. □

Next we extend Lemma 9.3 to infinite orthonormal families in Hilbert spaces.

Proposition 9.4 *Let $\{u_n\}_{n\in\mathbb{N}}$ be an orthonormal sequence in a Hilbert space H. Then for any $u \in H$, the series $\sum_{n=1}^{\infty}(u, u_n)u_n$ converges and*

$$\left\| u - \sum_{n=1}^{\infty} (u, u_n) u_n \right\|^2 = \|u\|^2 - \sum_{n=1}^{\infty} |(u, u_n)|^2 = \|u\|^2 - \left\| \sum_{n=1}^{\infty} (u, u_n) u_n \right\|^2. \tag{9.12}$$

Moreover $\sum_{n=1}^{\infty}(u, u_n)u_n$ coincides with the orthogonal projection of u on the closed subspace $\overline{span}\,(\{u_n\}_{n\in\mathbb{N}})$.

Proof. By the completeness of H, the series $\sum_{n=1}^{\infty} \alpha_n u_n$ converges. Passing to the limit as $m \to \infty$ in (9.6), we then get (9.12). □

The first equality in (9.12) obviously entails the **Bessel inequality**

$$\|u\|^2 \geq \sum_{n=1}^{\infty} |(u, u_n)|^2 \qquad \forall u \in H. \tag{9.13}$$

9.3 Hilbert bases

An orthonormal subset A of a Hilbert space H is called a **Hilbert basis** iff $H = \overline{\text{span}}\,(A)$. In this case A cannot be extended to any larger orthonormal subset of H; one then says that the orthonormal subset A is **complete.** [39] Notice that a Hilbert basis may be finite, countable or also uncountable.

Theorem 9.5 *Any Hilbert space has a Hilbert basis.*

Outline of the Proof. The argument may be based on the Zorn lemma or also the Hausdorff's maximal chain theorem. [Ex] □

Proposition 9.6 *Let J be any index set and $A := \{u_j : j \in J\}$ be an orthonormal subset of a Hilbert space H. The following properties are mutually equivalent:*
(i) A is a Hilbert basis;
(ii) A is maximal (w.r.t. the ordering by inclusion) among orthonormal subsets;
(iii) 0 is the only element of H orthogonal to A. [Ex]

*9.4 Unconditional convergence and sums

Let $\{u_n\}$ be a sequence in a normed space X. A series $\sum_{n=1}^{\infty} u_n$ is said to be **unconditionally convergent** iff $\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} u_{\pi(n)}$ for all permutations $\pi : \mathbb{N} \to \mathbb{N}$, that is, the convergence of the series does not depend on the order of its terms.

This may be extended by the notion of **sum** over any (nonempty) index set J, that is defined as follows. For any $\{u_j : j \in J\} \subset X$, we set $u := \sum_{j \in J} u_j$ iff, for any $\varepsilon > 0$, there exists a finite set $J_{u,\varepsilon} \subset J$ such that

$$\forall \text{ finite set } I \subset J, \qquad J_{u,\varepsilon} \subset I \quad \Rightarrow \quad \left\| \sum_{j \in I} u_j \right\| \leq \varepsilon. \tag{9.14}$$

In this case the set $\{u_j : j \in J\} \subset X$ is said to be **summable**.

Proposition 9.7 *A subset $\{u_i\}_{i \in I}$ of a normed space is* ***summable*** *iff*
(i) the set $J := \{i \in I : u_i \neq 0\}$ is at most countable, and
(ii) the series $\sum_{i \in J} u_i$ converges unconditionally. [Ex]

Let us next come back to Hilbert bases.

Proposition 9.8 *Let J be any index set and $A := \{u_j : j \in J\}$ be an orthonormal subset of a Hilbert space H. Then A is a Hilbert basis iff for any $u \in H$, there exists exactly one function $J \to \mathbb{K} : j \mapsto \alpha_j$ such that $u = \sum_{j \in J} \alpha_j u_j$ unconditionally. Moreover, if the latter property holds, then $J_u := \{j \in J : a_j \neq 0\}$ is at most countable. []*

[39] This should not be confused with the completeness of the space.

9.5 Fourier coefficients

Let $A := \{u_j : j \in J\}$ be an orthonormal subset of a Hilbert space H, and for any $u \in H$ set $\alpha_j := (u, u_j)$; thus, as we saw, $u = \sum_{j\in J} \alpha_j u_j$ (unconditionally). The α_js are called the (generalized) **Fourier coefficients** of u w.r.t. the orthonormal subset $\{u_j\}$. This denomination arose from and refers to the special case of the (separable) space $H = L^2(-\pi, \pi; \mathbb{C})$ and $u_k(x) = e^{ikx}/\sqrt{2\pi}$ for any $x \in\,]-\pi, \pi[$ and any $k \in \mathbb{Z}$, see ahead. By an obvious bijection between $\mathbb{Z}$ and $\mathbb{N}$, one may replace the indices $k \in \mathbb{Z}$ by $j \in \mathbb{N}$.

Next we provide more precise information about these coefficients.

Theorem 9.9 *Let $\{u_j\}_{j\in J}$ be an orthonormal subset of a Hilbert space H. The following properties are mutually equivalent:*

$$\{u_j\}_{j\in J} \text{ is a Hilbert basis;} \tag{9.15}$$

$$u = \sum_{j\in J} (u, u_j) u_j \qquad \forall u \in H \textit{ (Fourier expansion)}; \tag{9.16}$$

$$(u, v) = \sum_{j\in J} (u, u_j)\overline{(v, u_j)} \qquad \forall u, v \in H \textit{ (Parseval identity)}; \tag{9.17}$$

$$\|u\|^2 = \sum_{j\in J} |(u, u_j)|^2 \qquad \forall u \in H \textit{ (Parseval formula)}. \tag{9.18}$$

If J is countable, H is thus isometrically isomorphic to ℓ^2.

Proof. Let us assume (9.15). For any $u \in H$, by the Bessel inequality (9.12), $\sum_{\ell\in L} |(u, u_\ell)|^2 < +\infty$ for any finite set $L \subset J$. As we saw, the set $J_u := \{j \in J : (u, u_j) \neq 0\}$ is at most countable. By the final part of Proposition 9.4, $\sum_{j\in J_u} (u, u_j)u_j$ then coincides with the orthogonal projection of u on the closed subspace spanned by $\{u_j\}_{j\in J}$. By Proposition 9.8 this closed subspace coincides with the whole H. Therefore (9.15) entails (9.16).

For any finite set $L \subset J$, we have

$$\left(\sum_{j\in J} (u, u_j)u_j, \sum_{\ell\in L} (v, u_\ell)u_\ell\right) = \sum_{j\in J} (u, u_j)\overline{(v, u_j)} \qquad \forall u, v \in H;$$

(9.16) thus yields (9.17). Taking $v = u$ in (9.17) we get (9.18).

If (9.18) is fulfilled, then by (9.12) the closed subspace spanned by $\{u_j\}_{j\in J}$ coincides with H; (9.15) thus holds. □

9.6 Hilbert dimension

Proposition 9.10 *Any two Hilbert bases of the same Hilbert space have the same cardinality.*

This is called the **Hilbert dimension** (or the **orthonormal dimension**) of the space.

Proof. If one of the bases is finite, the other one is also finite, and the result is straightforward. Let us then assume that $\{u_j\}_{j\in J}$ and $\{v_\ell\}_{\ell\in L}$ are two infinite bases of a Hilbert space. For any $\ell \in L$, as we saw the set $J_\ell := \{j \in J : (v_\ell, u_j) \neq 0\}$ is at most countable.

Let us denote by card(A) the cardinality of any set A. Any $j \in J$ is element of some J_ℓ, as $\{v_\ell\}_{\ell\in L}$ is a Hilbert basis; hence card $(J) \le$ card $(\cup_{\ell\in L} J_\ell)$. Moreover card $(\cup_{\ell\in L} J_\ell) \le$ card (L), as each J_ℓ is at most countable and card (L) is infinite. Thus card $(J) \le$ card (L). By the symmetry of the argument, we infer that this is an equality. □

The Hilbert dimension may be either finite, or countable, or uncountable. Two Hilbert spaces are isometrically isomorphic iff they have the same dimension. [] The Hilbert dimension of a Hilbert space coincides with its Hamel dimension iff the space is finite-dimensional. [Ex]

9.7 Fourier series in L^2

The sequence of unit vectors $\{e_n\}$ is the canonical Hilbert basis of ℓ^2.

The linear space $L^2(-\pi,\pi)$ equipped with the inner product $(u,v) := \int_{-\pi}^{\pi} u(x)\overline{v(x)}\,dx$ is a complex Hilbert space.

Proposition 9.11 *The family $\{u_k(x) = e^{ikx}/\sqrt{2\pi}\}_{k\in\mathbb{Z}}$ is a Hilbert basis of $L^2(-\pi,\pi)$.*

Proof. By the Stone-Weierstrass theorem, $\{u_k\}_{k\in\mathbb{Z}}$ is dense in $C^0([-\pi,\pi])$. [] By the continuity of the canonic injection $C^0([-\pi,\pi]) \to L^2(-\pi,\pi)$, this family is dense in the latter space, too. □

The following formulae define the transform $L^2(-\pi,\pi) \to \ell^2 : f \mapsto \{\hat{f}_k\}_{k\in\mathbb{N}}$ and its inverse:

$$\hat{f}_k = (f, u_k) \quad \forall k \in \mathbb{Z}, \qquad f(x) = \sum_{k\in\mathbb{Z}} \hat{f}_k u_k(x) \quad \text{for a.e. } x \in \,]-\pi,\pi[, \tag{9.19}$$

the convergence of the latter series being understood in the sense of $L^2(-\pi,\pi)$. More explicitly, the two latter formulae read [40]

$$\begin{aligned} &\hat{f}_k = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(x) e^{-ikx}\,dx \qquad \forall k \in \mathbb{Z}, \\ &\lim_{m\to\infty} \int_{-\pi}^{\pi} \left| f(x) - \sum_{k=-m}^{k=m} \hat{f}_k u_k(x) \right|^2 dx = 0. \end{aligned} \tag{9.20}$$

The operator $L^2(-\pi,\pi) \to \ell^2 : f \mapsto \{\hat{f}_k\}$ is also called the *Fourier series transform in L^2.*

9.8 Overview

So far we have introduced three types of bases:

(i) Hamel bases for linear spaces: they exist for any space;

(ii) Schauder bases for separable Banach spaces: for some spaces they do not exist;

(iii) Hilbert bases for Hilbert spaces: they exist for any space.

A Schauder basis of a separable Hilbert space is a Hilbert basis iff it is orthonormal. In this case the Schauder basis is unconditional. [Ex]

We introduced the axioms of the inner product, and derived some basic properties, in particular the Cauchy-Schwarz inequality and the parallelogram identity. By means of the Cauchy-Schwarz inequality, we showed that a norm can be associated with any inner product. Thus any Hilbert space is also a Banach space. Conversely, because of the polarization identity, an inner product is associated with any norm which fulfills the parallelogram identity.

By means of the inner product, we defined the concepts of orthogonality and of orthogonal projection. The completeness entails the existence of the orthogonal projection on any (nonempty) closed convex subset, in particular on closed subspaces. Orthogonal projections are characterized as idempotent, self-adjoint (or equivalently, non expansive), linear operators of the space to itself. Orthogonal projections also provide a surjective isometric isomorphism between the

[40] A priori the L^2-convergence entails the a.e. convergence just for a suitable subsequence. In 1966 Carleson was able to prove the convergence of the whole sequence: a highly nontrivial result! On this basis in 2006 he was awarded the prestigious Abel prize.

antidual of any Hilbert space and the space itself (Riesz-Fréchet representation theorem). This entails that any Hilbert space is reflexive.

We then dealt with orthonormal systems of a Hilbert space and derived the Bessel inequality. We defined Hilbert bases, saw that any Hilbert space is endowed with such a basis, and defined the Hilbert dimension. Finally, we derived the Fourier expansion of any element of a Hilbert space w.r.t. to a Hilbert basis.

9.9 Exercises

1. Show that any orthonormal sequence in a Hilbert space H weakly vanishes.
2. Show that a Hilbert space is separable iff its Hilbert dimension is at most countable.
3. Show that two Hilbert spaces are isometrically isomorphic iff they have the same Hilbert dimension.
4. Show that any separable Hilbert space over the field $\mathbb{K}$ is isometrically isomorphic either to $\mathbb{K}^N$, for some $N \geq 0$, or to ℓ^2.
5. Let $\{e_n\}$ be the canonical basis of the Hilbert space ℓ^2, and V be the (dense) linear subspace spanned by $\{\sum_{n=1}^{\infty} e_n/n\} \cup \{e_n : n \geq 2\}$. Does an orthonormal Schauder basis exist in V?
6. = Let H be a Hilbert space and $L : H \to H$ be a linear mapping. Prove that if L is either symmetric (i.e., $(Lu, v) = (u, Lv)$ for any $u, v \in H$) or skew-symmetric (i.e., $(Lu, v) = -(u, Lv)$ for any $u, v \in H$), then it is continuous (this is the classical Hellinger-Toeplitz theorem).

   Hint: Remind the following result (one of the exercises of Section 2): Let X, Y be Banach spaces, and $L : X \to Y$ be linear. If $u_n \to u$ entails $Lu_n \rightharpoonup Lu$, then L is continuous.

10 Operators

10.1 Examples of Bounded Linear Operators

(i) For any matrix $A \in \mathbb{K}^{M,N}$, the associated linear mapping $L : \mathbb{K}^N \to \mathbb{K}^M$

$$(Lu)_j = \sum_{k=1}^{N} a_{jk} u_k \qquad 1 \leq j \leq M$$

obviously defines a bounded linear operator.

(ii) An **infinite matrix** $A = (a_{jk})$ defines a bounded linear mapping between sequence spaces through the formula

$$(Lu)_j = \sum_{k=1}^{\infty} a_{jk} u_k, \quad 1 \leq j < \infty. \tag{10.1}$$

For example, the estimate

$$\sum_{j=1}^{\infty} \left(\sum_{k=1}^{\infty} |a_{jk} u_k| \right)^2 \leq \left(\sum_{j,k=1}^{\infty} |a_{jk}|^2 \right) \sum_{k=1}^{\infty} |u_k|^2 \quad [Ex] \tag{10.2}$$

entails that $L \in \mathcal{L}(\ell^2)$ with $\|L\|^2 \leq \sum_{j,k} |a_{jk}|^2$, if this sum is finite. The latter condition, however, is not necessary: e.g., the unit matrix does not satisfy it. Indeed, in the diagonal case $(Lu)_k = \alpha_k u_k$, we have $L \in \mathcal{L}(\ell^p)$ iff $\|\alpha\|_\infty < \infty$. One may ask for conditions, in terms of the elements of A, which hold iff (10.1) defines a bounded linear mapping from ℓ^p to ℓ^q. However, no "useful" one is known in the case $1 < p, q < \infty$. []

(iii) The **right** (or **forward**) **shift** S_r and the **left** (or **backward**) **shift** S_l

$$(\tilde{S}_r u)_k = u_{k-1}, \quad (\tilde{S}_\ell u)_k = u_{k+1}, \tag{10.3}$$

are most naturally defined on the space of doubly infinite sequences $\{u_k\}_{k\in\mathbb{Z}}$; obviously these are isometries on $\ell^p_{\mathbb{K}}(\mathbb{Z})$ for any $p \in [1, \infty]$. [41] Most often one deals with unilateral sequences $u = (u_1, u_2, \dots)$, and sets

$$S_r(u_1, u_2, \dots) = (0, u_1, u_2, \dots), \quad S_\ell(u_1, u_2, \dots) = (u_2, u_3, \dots).$$

These operators belong to $\mathcal{L}(X)$ for $X = \ell^p$ $(:= \ell^p_{\mathbb{K}}(\mathbb{N}))$, but they are no longer isomorphisms. Notice that S_r is injective but not surjective, and S_ℓ is surjective but not injective; $S_\ell \circ S_r = I$ but $S_r \circ S_\ell \neq I$.

(iv) For any $p \in [1, \infty]$, if a is a bounded measurable function on a measure space $(A, \mathcal{A}, \mu)$, then the **multiplication operator** defined by

$$(Lu)(x) = a(x)u(x) \qquad \text{for a.e. } x \in A$$

is an operator $L \in \mathcal{L}(L^p(A))$ and $\|L\| = \|a\|_\infty$. Similarly, if A is also a compact metric space and $a \in C^0(A)$, then $L \in \mathcal{L}(L^p(A))$ and $\|L\| = \|a\|_\infty$.

(v) Let $(A, \mathcal{A}, \mu)$ and $(B, \mathcal{B}, \nu)$ be two σ-finite measure spaces, $k \in L^2(A \times B)$, and set

$$(Lu)(x) = \int_B k(x, y)u(y)\, d\mu(y) \qquad \text{for a.e. } x \in A, \forall u \in L^2(B). \tag{10.4}$$

By the theorems of Tonelli and Fubini and the Hölder inequality, Lu is an a.e. well-defined and measurable function, and

$$\int_A \left| \int_B k(x, y)u(y)\, d\nu(y) \right|^2 d\mu(x) \leq \int_A \int_B |k(x, y)|^2\, d\nu(y)\, d\mu(x) \cdot \int_B |u(y)|^2\, d\nu(y).$$

Thus $L \in \mathcal{L}(L^2(B); L^2(A))$, and

$$\|L\|_{\mathcal{L}(L^2(B);L^2(A))} \leq \iint_{A\times B} |k(x, y)|^2\, d\mu(y)\, d\nu(x).$$

The function k is called the **kernel** of the **integral operator** L.

If $A = B = [a, b]$ and $\mu = \nu$ is the Lebesgue measure, then the operators

$$(L_1 u)(x) = \int_a^b k(x, y)u(y)\, dy, \quad (L_2 u)(x) = \int_a^x k(x, y)u(y)\, dy \quad \forall x \in [a, b]$$

are respectively called **Fredholm** and **Volterra integral operators**.

*10.2 Adjoint of unbounded operators

Adjoint operators are illustrated e.g. in [Bz, chap. 2]. That definition is extended to unbounded operators (i.e., linear operators $L : D(L) \subset X \to Y$), [42] as follows, assuming that $\overline{D(L)} = X$. For any $f \in Y'$, let us say that $f \in D(L^*)$ iff the functional $D(L) \to \mathbb{C} : u \mapsto \langle Lu, f\rangle$ can be extended to a functional $L^* f \in X'$. Thus

$$\langle Lu, f\rangle = \langle u, L^* f\rangle \qquad \forall u \in D(L), \forall f \in D(L^*). \tag{10.5}$$

It is not difficult to prove the following result.

[41] Which norm on $\ell^p_{\mathbb{K}}(\mathbb{Z})$ makes it isometrically isomorphic to $\ell^p_{\mathbb{K}}(\mathbb{N})$? (The norm is not unique.) [Ex]

[42] So what are named unbounded operators are actually *either bounded or unbounded* linear operators.

Theorem 10.1 *If X, Y are Banach spaces and $L : D(L) \subset X \to Y$ is a linear operator, then*

$$L \text{ is bounded} \quad \Leftrightarrow \quad L^* \text{ is bounded} \quad \Rightarrow \quad \|L\|_{\mathcal{L}(X,Y)} = \|L^*\|_{\mathcal{L}(Y',X')}, \tag{10.6}$$

$$\exists L^{-1} \text{ bounded} \quad \Leftrightarrow \quad \exists (L^*)^{-1} \text{ bounded} \quad \Rightarrow \quad \|L^{-1}\|_{\mathcal{L}(Y,X)} = \|(L^*)^{-1}\|_{\mathcal{L}(X',Y')}, \tag{10.7}$$

$$\overline{D(L)} = X \quad \Rightarrow \quad L^* \text{ is uniquely defined, } \mathrm{graph}(L^*) \text{ is closed (in } Y' \times X'), \tag{10.8}$$

$$D(L^*) \text{ is closed} \quad \Rightarrow \quad L^* \in \mathcal{L}(D(L^*), X'). \tag{10.9}$$

The final statement follows from the previous one. Indeed, if $D(L^*)$ is closed, then graph(L^*) is closed. [Ex] This entails (10.9), since any unbounded operator with closed domain and closed graph is actually bounded, because of the Closed Graph Theorem.

We shall say that a linear operator $L : D(L) \subset X \to Y$ is *bounded below* iff there exists $C > 0$ such that $\|Lv\|_Y \geq C\|v\|_X$ for any $v \in D(L)$. In (2.11) we already defined the kernel and the range of a linear operator.

Theorem 10.2 *If X, Y are Banach spaces and $L \in \mathcal{L}(X, Y)$, then*

$$\mathcal{N}(L) = {}^{\perp}\mathcal{R}(L^*), \qquad \mathcal{N}(L^*) = \mathcal{R}(L)^{\perp}, \tag{10.10}$$

$$\mathcal{R}(L) \text{ is closed} \quad \Leftrightarrow \quad \mathcal{R}(L^*) \text{ is closed} \quad \text{(Banach's Closed-Range Theorem), []} \tag{10.11}$$

$$\begin{aligned} &L \text{ is bounded below} \quad \Leftrightarrow \quad L \text{ is injective, } \mathcal{R}(L) \text{ is closed} \\ &\Leftrightarrow \quad L^* \text{ is surjective} \quad \Leftrightarrow \quad \exists L^{-1} \in \mathcal{L}(\mathcal{R}(L), X), \end{aligned} \tag{10.12}$$

$$L \text{ is bijective} \quad \Leftrightarrow \quad L \text{ is bounded below, } \mathcal{R}(L) \text{ is dense.} \tag{10.13}$$

The argument of Banach's Closed-Range Theorem (10.11) is nontrivial, see e.g. [Bz, chap. 2]; once it has been established, the other statements follow without difficulty. [Ex]

From (10.10) one may infer that

$$\mathcal{N}(L)^{\perp} = \overline{\mathcal{R}(L^*)}^{w*}, \qquad {}^{\perp}\mathcal{N}(L^*) = \overline{\mathcal{R}(L^*)}. \tag{10.14}$$

The dual of the statements (10.12) and (10.13) also hold:

$$\begin{aligned} &L^* \text{ is bounded below} \quad \Leftrightarrow \quad L^* \text{ is injective, } \mathcal{R}(L^*) \text{ is closed} \\ &\Leftrightarrow \quad L \text{ is surjective} \quad \Leftrightarrow \quad \exists (L^*)^{-1} \in \mathcal{L}(\mathcal{R}(L^*), X), \end{aligned} \tag{10.15}$$

$$L^* \text{ is bijective} \quad \Leftrightarrow \quad L^* \text{ is bounded below, } \mathcal{R}(L^*) \text{ is dense.} \tag{10.16}$$

The whole theorem holds also for unbounded operators with dense domain and closed graph, but in that case the arguments are more demanding; see e.g. [Bz, chap. 2].

Density of the domain and closedness of the graph are typical hypotheses in the theory of unbounded operators. Note that the density of the domain may be retrieved just by restricting L to the closure of its domain. On the other hand, if the graph of L is not closed, in several cases the closure of this graph is the graph of a single-valued (hence linear) operator, $\bar{L}$. This operator is called the *closure* of L, since it is the smallest closed operator extending L; L is then said to be a *closable* operator. This notion is especially relevant for unbounded operators on Hilbert space. [43]

Theorem 10.3 *Let H be a Hilbert space. If $L : D(L) \subset H \to H$ is a linear operator with dense domain, then:*

(i) L is closable iff L^* has dense domain.

(ii) If L is closable, then $\bar{L} = (L^*)^*$ (=: L^{**}) and $(\bar{L})^* = L^*$.

(iii) If $D(L^*)$ is closed, then $L^* : D(L^*) \to H$ is continuous.

[43] Unbounded self-adjoint operators are at the basis of the formulation of quantum mechanics that was given by John von Neumann (born 1903) in 1927-28.

10.3 Exercises

1. (i) Prove (10.8) and (10.9) for a linear and bounded operator.
   (ii) Prove (10.12) and (10.13) (for a linear and bounded operator).

2. Let X, Y be Banach spaces. Any $L \in \mathcal{L}(X, Y)$ has closed graph and closed kernel, but the range $L(X_1)$ need not be closed (otherwise the Closed-Range Theorem would make no sense...). Give an example of $L \in \mathcal{L}(X, Y)$ with nonclosed range.

3. * Let X, Y be Banach spaces. Show that if $L \in \mathcal{L}(X, Y)$ and L^* are bounded below, then there exists $L^{-1} \in \mathcal{L}(Y, X)$.

   Hint: As L is bounded below, $R(L)^\perp = N(L^*) = \{0\}$...

10.4 Compact operators

Throughout this section X and Y will denote Banach spaces over $\mathbb{K}$.

A linear and continuous operator $K : X \to Y$ is called **compact** (or **completely continuous**) iff it maps any bounded subset of X to a relatively compact subset of Y, or equivalently iff $K(B_X)$ (the image of the closed unit ball) is relatively compact in Y. We shall denote the set of all compact operators by $\mathcal{K}(X; Y)$, or $\mathcal{K}(X)$ if $X = Y$. Obviously, $K : X \to Y$ is compact iff for every bounded sequence $\{u_n\}$ in X, the sequence $\{Ku_n\}$ has a convergent subsequence.

Proposition 10.4 *The composition $M \circ L$ of two linear continuous operators is compact if either M or L is compact. (That is, $\mathcal{K}(X; Y)$ is a two-sided ideal in the Banach algebra $\mathcal{L}(X; Y)$.)*

Proof. This holds since continuous operators map bounded sets to bounded sets, and relatively compact sets to relatively compact sets. □

Proposition 10.5 *$\mathcal{K}(X; Y)$ is a closed subspace of the Banach space $\mathcal{L}(X; Y)$.*

Proof. It is promptly seen that $\mathcal{K}(X; Y)$ is a linear subspace; hence it suffices to prove that the limit of any uniformly converging sequence of compact operators is a compact operator.

Let $L \in \mathcal{L}(X; Y)$ and $\{K_m\}$ be a sequence in $\mathcal{L}(X; Y)$ such that $K_m \to L$. Let $\{u_n\}$ be a bounded sequence in X, $\|u_n\| \le C$. It suffices to prove that $\{Lu_{n_k}\}$ is Cauchy in Y for some subsequence. To this purpose, by a diagonalization procedure one may select $\{u_{n_k}\}$ such that $\{K_m u_{n_k}\}$ is convergent in Y for all $m \in \mathbb{N}$. Let now $\varepsilon > 0$ be given. Choose m with $\|L - K_m\| \le \varepsilon$, and h_0 with $\|K_m u_{n_k} - K_m u_{n_j}\| < \varepsilon$ for all $k, j \ge h_0$. This entails that, for all $k, j \ge h_0$,

$$\|Lu_{n_k} - Lu_{n_j}\| \le \|K_m u_{n_k} - K_m u_{n_j}\| + \|L - K_m\|(\|u_{n_k}\| + \|u_{n_j}\|) \le (1 + 2C)\varepsilon.$$

So $\{Lu_{n_k}\}$ indeed is a Cauchy sequence in Y. □

Theorem 10.6 (*Schauder*) *An operator $L \in \mathcal{L}(X; Y)$ is compact iff its adjoint L^* is compact.*

* *Proof.* Let L be compact, then $A = \overline{L(B_X)}$ is compact. Let $\{g_n\}$ be a sequence in Y' with $\|g_n\| \le 1$. Since $|g_n(v) - g_n(w)| \le \|v - w\|$ for all $v, w \in A$, the sequence $\{g_n|_A\}$ is bounded and equicontinuous in $C^0(A)$. By the Ascoli-Arzelà theorem, some subsequence $\{g_{n_k}\}$ then converges uniformly on A, and thus is Cauchy (w.r.t. the uniform norm). As

$$\|L^* g_{n_k} - L^* g_{n_j}\|_{X'} = \sup_{x \in B_X} \langle g_{n_k} - g_{n_j}, Lx \rangle \le \|g_{n_k} - g_{n_j}\|_{C^0(A)} \|L\|_{\mathcal{L}(X,Y)} \quad \forall k, j,$$

the subsequence $\{L^* g_{n_k}\}$ is Cauchy and hence convergent in X'. Thus L^* is compact.

If conversely L^* is compact, then so is $L^{**} : X'' \to Y''$. Since $J_Y(L(B_X)) = L^{**}(J_X(B_X))$, the set $J_Y(L(B_X))$ is relatively compact in Y''. As J_Y is an isometry we conclude that $L(B_X)$ is relatively compact in Y. □

10.5 Examples

(i) If X has finite dimension, then every $L \in \mathcal{L}(X;Y)$ is compact.

(ii) The shift operators S_r and S_ℓ are not compact on ℓ^p, since $S_\ell \circ S_r = I$ is not compact.

(iii) Let A be a Euclidean set with nonempty interior and $X = C^0_b(A)$. The multiplication operator

$$(L_a u)(x) = a(x)u(x) \qquad \forall u \in X, \text{for a given } a \in C^0_b(A)$$

is compact only if $a \equiv 0$ identically in A. [Ex] An analogous result holds for $X = L^p(A)$.

(iv) Depending on the properties of its kernel k, the integral operator

$$(Lu)(x) = \int_A k(x,y)u(y)\,d\nu(y) \tag{10.17}$$

not only belongs to $\mathcal{L}(X;Y)$ for suitable function spaces X and Y, but actually is compact. Consider for example the case $X = L^2(A_1;\mu)$, $Y = L^2(A_2;\nu)$ with σ-finite measures μ and ν, and with $k \in L^2(A_1 \times A_2; \mu \otimes \nu)$.

If k has the product form $k(x,y) = g(x)h(y)$ with $g \in L^2(A_2)$ and $h \in L^2(A_1)$, then

$$(Lu)(x) = g(x)\int_{A_1} h(y)u(y)\,d\nu(y) \qquad \text{for } \mu\text{-a.e. } x \in A_2.$$

Thus $\mathcal{R}(L)$ equals the one-dimensional subspace spanned by g; in this case L is thus compact.

For the general case, it turns out that for any $\varepsilon > 0$ there is a $k_\varepsilon \in L^2(A_1 \times A_2)$, which is a finite linear combination of kernels in product form and satisfies $\|k - k_\varepsilon\|_{L^2(A_1\times A_2)} \le \varepsilon$. [Ex] The corresponding integral operator L_ε has finite rank [44] and satisfies $\|L - L_\varepsilon\| \le \|k - k_\varepsilon\|_{L^2(A_1\times A_2)} \le \varepsilon$. As $\mathcal{K}(X;Y)$ is a closed subspace of $\mathcal{L}(X;Y)$, L is thus compact.

(v) As another example of operator of the form (10.17), let us consider the case where $X = C^0(A_1)$ and $Y = C^0(A_2)$, with A_1 and A_2 compact subsets of $\mathbb{R}^N$ and $\mathbb{R}^M$ (resp.) equipped with the Lebesgue measure, and with $k \in C^0(A_1 \times A_2)$. Since k is uniformly continuous, the image $L(B_1)$ of the closed unit ball B_1 in $C^0(A_1)$ is equicontinuous, hence relatively compact in $C^0(A_2)$ by the Ascoli-Arzelà theorem; so L is compact in this case, too.

10.6 Exercises

1. Let X, Y be Banach spaces, and let a linear operator $L : X \to Y$ map the closed unit ball B_X to a relatively compact subset of Y. Is then L continuous?

2. = Let X, Y be Banach spaces, and let a linear operator $L : X \to Y$ map the closed unit ball B_X to a relatively compact subset of Y. Is then L continuous?

3. Let us fix any $f \in C^1_b(\mathbb{R}^2)$ and set $[T(v)](x) = \int_0^x f(x,y)v(y)\,dy$ for any x.

   (i) Is T a bounded operator $C^0([0,1]) \to C^1([0,1])$?
   (ii) Is T a compact operator $C^0([0,1]) \to C^0([0,1])$?
   * (iii) Is T a compact operator $C^0([0,1]) \to C^1([0,1])$?
   (iv) Is T a bounded operator $C^0_b(\mathbb{R}) \to C^0_b(\mathbb{R})$?

4. Let X, Y, Z be Banach spaces and $L \in \mathcal{L}(X,Y)$, $M \in \mathcal{L}(Y,Z)$. Prove that if either of these operators is compact, then their composition ML is also compact.

[44] The English terms *rank* and *range* respectively correspond to *rango* e *insieme immagine.*

5. Let X, Y be Banach spaces.

   Characterize the class of operators $L \in \mathcal{L}(X,Y)$ that are compact and bijective.

6. (i) Are the inclusions among the spaces ℓ^p $(1 \le p \le +\infty)$ compact?
   (ii) Are the inclusions among the spaces $L^p(0,1)$ $(1 \le p \le +\infty)$ compact?

7. Are the canonical injections $C^{k+1}([0,1]) \to C^k([0,1])$ $(k \in \mathbb{N})$ compact?

8. Is the canonical injection $C_b^1(\mathbb{R}) \to C_b^0(\mathbb{R})$ compact?

9. Is the canonical injection $C^0([0,1]) \to L^p(0,1)$ compact for some $1 \le p \le +\infty$?

10. Let $1 \le p < +\infty$. Are the canonical injections $\ell^p \to c_0$, $\ell^p \to c$, $\ell^p \to \ell^\infty$ compact?

11. = * Let X, Y be two Banach spaces, $L \in \mathcal{L}(X,Y)$, and denote by B_X the closed unit ball of X.

    (i) Show that if X is reflexive, then $L(B_X)$ is closed.
    Hint: Apply the Banach-Alaoglu theorem to converging sequences in B_X...

    (ii) Show that if X is reflexive and L is compact, then $L(B_X)$ is compact.
    * (iii) Check that if $X = Y = C^0([0,1])$ and $(Lu)(x) = \int_0^x u(t)\,dt$ for any x, then $L(B_X)$ is not closed.

    (This exercise has been taken from [Bz] p. 171.)

12. = Let X, Y be two Banach spaces, with X of infinite dimension, and let $L \in \mathcal{L}(X,Y)$ be compact. Show that there exists a sequence $\{u_n\}$ in X such that $\|u_n\| = 1$ for any n and $Lu_n \to 0$ in Y.

    Hint: Use the Riesz lemma...

    (This exercise has been taken from [Bz] p. 171.)

13. = Let X and Y be two Banach spaces and $L \in \mathcal{L}(X;Y)$. Show that:

    (i) If L is compact then L maps every weakly converging sequence $\{u_n\}$ in X to a strongly converging sequence $\{Lu_n\}$ in Y. (The latter property is named *complete continuity).*

    (ii) The converse is true if X is reflexive, but not in general.

    Hint: As a counterexample, consider the identity map in ℓ^1...

14. = Show that the adjoint of an integral operator with kernel $k(x,y)$ is also an integral operator, and identify its kernel.

15. = * Let X and Y be two Banach spaces and $L \in \mathcal{L}(X;Y)$ be compact. Show that if $\mathcal{R}(L)$ is closed then it is finite-dimensional.

    Hint: $\mathcal{R}(L)$ is complete, as it is closed. By the open mapping theorem, $L : X \to \mathcal{R}(L)$ then maps closed sets to compact sets. As L is compact, it then maps any bounded closed ball to a compact set. $\mathcal{R}(L)$ is thus a countable union of compact sets...

10.7 The Fredholm-Riesz-Schauder Theory

An important result in linear algebra states that a linear operator between finite-dimensional linear spaces is injective iff it is surjective. This fails in infinite dimension; e.g., in any ℓ^p the right shift S_r is injective but not surjective, and the left shift S_ℓ is surjective but not injective. In this section we shall see how the above result may be extended to a relevant class of infinite-dimensional linear and continuous operators (This extension is known as the theorem of the *Fredholm alternative.)*

We say that a linear operator $L : X \to Y$ has **finite rank** iff its range $\mathcal{R}(L)$ has finite dimension. Any finite-rank operator acting between Banach spaces X, Y is clearly compact.

It is promptly seen that the family of finite-rank operators is not closed in $\mathcal{L}(X,Y)$, at variance with the family of compact operators. Although there are compact operators that cannot be represented as the limit in norm of any sequence of finite-rank operators, [45] the properties of compact operators make the former family a natural extension of finite-rank operators, as we shall see in this section.

A celebrated theorem of F. Riesz extends some known properties of matrices (namely, of linear operators in Euclidean spaces) to compact perturbations of linear operators in Banach spaces. As a preliminary step, first we illustrate this result for finite-rank operators.

By Proposition 2.18 and as $\mathcal{N}(I-L) \subset \mathcal{R}(L)$, if X is a linear space and $L : X \to X$ is a linear mapping of finite rank, then

$$(i)\ \mathcal{N}(I-L) \text{ has finite dimension,} \tag{10.18}$$

$$(ii)\ \mathcal{R}(I-L) \text{ has finite codimension.} \tag{10.19}$$

The next classical result extends these properties to compact perturbations of the identity in a Banach space X.

Theorem 10.7 (*F. Riesz) Let X be a Banach space and $K : X \to X$ be a compact operator. Then*

$$(i)\ \mathcal{N}(I-K) \textit{ has finite dimension,} \tag{10.20}$$

$$(ii)\ \mathcal{R}(I-K) \textit{ is closed and has finite codimension,} \tag{10.21}$$

$$(iii)\ \dim(\mathcal{N}(I-K)) = \dim(\mathcal{N}(I-K^*)), \tag{10.22}$$

$$(iv)\ \mathcal{N}(I-K) = \{0\} \ \Leftrightarrow\ \mathcal{R}(I-K) = X \tag{10.23}$$

$$\textit{(i.e., } I-K \textit{ in injective iff it is surjective).} \tag{10.24}$$

Partial Proof. The restriction of K to $\mathcal{N}(I-K)$ coincides with the identity and is compact; this yields (10.20). The proof of parts (ii) and (iii) is more technical, and is here omitted. For part (iv) see below. □

Remark. This theorem also applies to operators of the form $A-K$ (in place of $I-K$), for any linear continuous isomorphism $A : X \to X$ and any compact operator $K : X \to X$. Actually, $A^{-1}K$ is also compact, so the theorem holds for $I - A^{-1}K$. Therefore it also holds for $A(I - A^{-1}K) = A - K$.

Corollary 10.8 *Let X be a Banach space and $K : X \to X$ be a compact operator. The thesis of the Riesz Theorem 10.7 then holds also for K^*. Moreover,*

$$\mathcal{R}(I-K) = {}^{\perp}[\mathcal{N}(I-K^*)], \tag{10.25}$$

and dually $\mathcal{R}(I-K^*) = [\mathcal{N}(I-K)]^{\perp}$ if X is reflexive.

Proof. For the first statement it suffices to notice that by the Schauder theorem the operator $K^* : X' \to X'$ is also compact, and to apply the Riesz theorem.

(10.25) stems from the closedness of $\mathcal{R}(I-K)$ and from the known formula $\mathcal{N}(I-K^*) = [\mathcal{R}(I-K)]^{\perp}$ (see e.g. [Br] p. 66). The final statement follows from the dual argument. □

On the basis of (10.25), part (iv) of Theorem 10.7 is easily derived from part (iii) of the same theorem. [Ex]

[45] A highly nontrivial counterexample was found by a Enflo (a counterexample hunter).

10.8 The Fredholm Alternative

The statement (10.23) expresses the Fredholm alternative: $I-K$ is injective iff it is surjective. Here we detail this issue for problems of the form

$$\text{for a prescribed } b \in X, \text{ find } u \in X \text{ such that} \quad u - Ku = b. \tag{10.26}$$

The next statement directly follows from the Riesz Theorem 10.7, and expresses several basic properties of the space of solutions, that carry over from the finite-dimensional to the infinite-dimensional setup.

Corollary 10.9 (*Fredholm alternative) Let X be a Banach space and $K : X \to X$ be a compact operator. Then one (and only one) of the following alternatives holds:*

> ***either (i)*** *the equation $u - Ku = b$ has a unique solution $u \in X$ for any $b \in X$,*
>
> ***or (ii)*** *the homogeneous equation $u - Ku = 0$ has a nontrivial solution $u \in X$, and there exists $b \in X$ such that the equation $u - Ku = b$ has no solution.*

In the second case, the inhomogeneous equation $u - Ku = b$ is solvable iff $f(b) = 0$ for all solutions $f \in X'$ of the homogeneous adjoint equation $(I - K^*)f = 0$.

Proof. The dichotomy between the cases (i) and (ii) directly follows from (10.23).
The final statement of the corollary stems from (10.25). □

Remarks. (i) The two cases of the dichotomy respectively correspond to

(1) $\mathcal{N}(I-K) = \{0\}$ and $\mathcal{R}(I-K) = X$, that is, $\dim(\mathcal{N}(I-K)) = \text{codim}(\mathcal{R}(I-K)) = 0$, and

(2) $1 \le \dim(\mathcal{N}(I-K)) = \text{codim}(\mathcal{R}(I-K)) = \dim(\mathcal{N}(I-K^*)) < +\infty$.

(iii) The equality $\dim(\mathcal{N}(I-K)) = \text{codim}(\mathcal{R}(I-K))$ means that the (finite) number of linearly independent solutions $u \in X$ of $u - Ku = 0$ equals the number of linearly independent constraints that define $\mathcal{R}(I-K)$. The latter set indeed coincides with the subspace of the elements of X that fulfill a finite number of linearly-independent linear equations: these are the constraints.

(ii) Let us consider the problems "$u \in X$, $u - Ku = 0$" and "$f \in X'$, $f - K^*f = 0$". The equality $\dim(\mathcal{N}(I-K)) = \dim(\mathcal{N}(I-K^*))$ means that these two problems have the same (finite) number of linearly independent solutions.

(iv) We already pointed out that the Riesz Theorem 10.7 applies to all operators of the form $A - K$, for any linear and continuous isomorphism $A : X \to X$ and any compact operator $K : X \to X$. The same then applies to the Fredholm alternative.

(v) If K is a compact operator is a Hilbert space H, the Riesz Theorem 10.7 may be reformulated as follows:

$$\dim(\mathcal{N}(I-K)) = \dim(\mathcal{N}(I-K^*)) < +\infty, \tag{10.27}$$

$$H = \mathcal{N}(I-K) \oplus \mathcal{R}(I-K^*).[] \tag{10.28}$$

11 Introduction to Spectral Analysis

11.1 Spectrum of linear and continuous operators

Let X be a Banach space. The **resolvent set** $\rho(L)$ of any $L \in \mathcal{L}(X)$ is defined as the set of the $\lambda \in \mathbb{K}$ such that $\lambda I - L$ has an inverse in $\mathcal{L}(X)$; this inverse is called the **resolvent operator**. This holds iff $\lambda I - L$ is bijective, as in this case $(\lambda I - L)^{-1}$ exists and is continuous, because of the open mapping principle. [46] Any $\lambda \in \rho(L)$ is called a **regular point**. For any $\lambda \in \rho(L)$, the resolvent operator $R_L(\lambda) := (\lambda I - I)^{-1}$ is thus a Banach space isomorphism on X.

The spectrum $\sigma(L)$ is defined as $\mathbb{K} \setminus \rho(L)$, and may be decomposed as follows. The **point spectrum** $\sigma_p(L)$ consists of all $\lambda \in \mathbb{K}$ such that $\lambda I - L$ is not injective. Thus $\lambda \in \sigma_p(L)$ iff $Lu = \lambda u$ for some $u \neq 0$; in this case λ is called an **eigenvalue** of L. (Note that for $\mathbb{K} = \mathbb{R}$, the point spectrum includes only the real eigenvalues of L.) For $\lambda \in \sigma_p(L)$, the subspace $\mathcal{N}(\lambda I - L)$ is called the **eigenspace** belonging to λ. Its nonzero elements are named **eigenvectors** belonging to λ.

The set $\sigma(L) \setminus \sigma_p(L)$ is named the **essential spectrum,** is denoted by $\sigma_e(L)$, and is divided into two parts:

(i) The **continuous spectrum** $\sigma_c(L)$ consists of all $\lambda \in \sigma(L) \setminus \sigma_p(L)$ for which $\mathcal{R}(\lambda I - L)$ is dense in X. (This entails that $(\lambda I - I)^{-1}$ is unbounded, since otherwise this operator would be extendable to the whole X, and thus $\lambda \in \rho(L)$.)

(ii) The **residual spectrum** $\sigma_r(L)$ is formed by the $\lambda \in \sigma(L) \setminus \sigma_p(L)$ for which $\mathcal{R}(\lambda I - L)$ is not dense in X.

Thus obviously

$$\sigma(L) = \sigma_p(L) \cup \sigma_e(L) = \sigma_p(L) \cup \sigma_c(L) \cup \sigma_r(L), \tag{11.1}$$

and these unions are disjoint. (Different terminologies may also be found in the literature.)

Proposition 11.1 * *Let X be a Banach space. For the adjoint L^* of any $L \in \mathcal{L}(X)$,*

$$\sigma(L^*) = \sigma(L), \tag{11.2}$$

$$\sigma_r(L) \subset \sigma_p(L^*) \subset \sigma_r(L) \cup \sigma_p(L). \tag{11.3}$$

* *Proof.* For any $\lambda \in \mathbb{K}$, $\lambda I - L^* = (\lambda I - L)^*$ is invertible iff $\lambda I - L$ is invertible; [Ex] hence $\rho(L^*) = \rho(L)$, which is tantamount to (11.2). Let us now remind that for any nonempty set $A \subset X$, ${}^\perp(A^\perp) = \overline{\text{span}(A)}$. [Ex] As $\mathcal{N}(\lambda I - L^*) = \mathcal{R}(\lambda I - L)^\perp$, we then have

$${}^\perp\mathcal{N}(\lambda I - L^*) = \overline{\mathcal{R}(\lambda I - L)}. \tag{11.4}$$

On the other hand, by definition of σ_r and σ_p,

$$\begin{aligned} \lambda \in \sigma_p(L^*) \quad &\Leftrightarrow \quad {}^\perp\mathcal{N}(\lambda I - L^*) \neq X, \\ \lambda \in \sigma_r(L) \quad &\Rightarrow \quad \overline{\mathcal{R}(\lambda I - L)} \neq X, \\ \overline{\mathcal{R}(\lambda I - L)} \neq X \quad &\Rightarrow \quad \lambda \in \sigma_r(L) \cup \sigma_p(L). \end{aligned} \tag{11.5}$$

The four latter displayed formulae entail (11.3). □

Two operators $L_1, L_2 \in \mathcal{L}(X)$ are called **similar** iff there exists an invertible operator $A \in \mathcal{L}(X)$ such that $L_1 = A^{-1}L_2A$.

Theorem 11.2 (*Spectral invariance) Let X be a Banach space. Similar operators have the same spectrum. In general the converse fails. [Ex]*

[46] Notice that $(\lambda I - L)^{-1} \in \mathcal{L}(X)$ whenever $\mathcal{R}(\lambda I - L)$ is dense in X and $(\lambda I - L)^{-1} : \mathcal{R}(\lambda I - L) \to X$ is bounded.

The converse fails even for (2×2)-matrices: e.g., any nilpotent (2×2)-matrix has a (double) vanishing eigenvalue (even if it is not similar to the null matrix).

The parts of the spectrum, namely $\sigma_p, \sigma_c, \sigma_r$, as well as the dimension of each eigenspace (i.e., the subspace of the eigenvectors associated to a same eigenvalue) are also invariant in each class of similar operators. [Ex]

Theorem 11.3 (*Spectral mapping theorem*) *Let X be a Banach space. For any polynomial p and any $L \in \mathcal{L}(X)$, $\sigma(p(L)) = p(\sigma(L)$. []*

Remarks. More generally, the latter theorem and several others concerning the space $\mathcal{L}(X)$ and spectral properties hold in Banach algebras with unit.

11.2 Examples

(i) As we know, a linear operator on a finite-dimensional space X is injective iff it is surjective. Therefore

$$\dim(X) < +\infty \quad \Rightarrow \quad \sigma(L) = \sigma_p(L) \quad (\text{i.e., } \sigma_c(L) \cup \sigma_r(L) = \emptyset) \quad \forall L \in \mathcal{L}(X).$$

In the analysis of the next example we use the following result. For any $L \in \mathcal{L}(X)$, let us first define its *spectral radius:* $r(L) := \sup\{|\lambda| : \lambda \in \sigma(L)\}$.

Theorem 11.4 *For any $L \in \mathcal{L}(X)$,*
(i) the spectrum $\sigma(L)$ is a nonempty compact subset of $\mathbb{C}$;
(ii) $r(L) \leq \|L\|$ (spectral radius theorem). []

* (ii) Let us analyse the spectrum of the shift operators $S_r, S_\ell \in \mathcal{L}(\ell^p_{\mathbb{K}}(\mathbb{N}))$ for any p, with the aid of Proposition 11.1. Let us first assume that $1 \leq p < +\infty$. We have $(\lambda I - S_\ell)u = 0$ iff $\lambda u_k = u_{k+1}$ for all k, that is, iff u is a multiple of $(1, \lambda, \lambda^2, \dots)$, and therefore $\sigma_p(S_\ell) = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$. Since $\|S_\ell\| = 1$ and $\sigma(L)$ is closed, it follows that $\sigma(S_\ell) = \{\lambda : |\lambda| \leq 1\}$.

Let us now come to S_r. As $S_r = S'_\ell$, $\sigma(S_r) = \sigma(S_\ell)$ by (11.2). Now $(\lambda I - S_r)u = 0$ iff $\lambda u_1 = 0$ and $\lambda u_k = u_{k-1}$ for $k > 1$, that is, iff $u = 0$; therefore $\sigma_p(S_r) = \emptyset$. As $S_r = S'_\ell$ and $S_\ell = S'_r$, by (11.3) it follows that $\sigma_r(S_r) = \sigma_p(S_\ell)$ and $\sigma_r(S_\ell) \subset \sigma_p(S_r) = \emptyset$; thus $\sigma_r(S_\ell) = \emptyset$. Moreover, as $\sigma_c = \sigma \setminus (\sigma_p \cup \sigma_r)$, we get $\sigma_c(S_\ell) = \sigma_c(S_r) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ In conclusion, for $1 \leq p < +\infty$ we have thus seen that

$$\begin{aligned} &\sigma_r(S_\ell) = \sigma_p(S_r) = \emptyset, \\ &\sigma_p(S_\ell) = \sigma_r(S_r) = \{\lambda \in \mathbb{C} : |\lambda| < 1\}, \\ &\sigma_c(S_\ell) = \sigma_c(S_r) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}. \end{aligned} \tag{11.6}$$

For $p = \infty$, a similar analysis shows that $\sigma_p(S_\ell) = \sigma(S_\ell) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$ and $\sigma_p(S_r) = \emptyset$. By (11.3) then

$$\begin{aligned} &\sigma_p(S_\ell) = \sigma_r(S_r) = \{\lambda : |\lambda| \leq 1\}, \\ &\sigma_r(S_\ell) = \ \sigma_c(S_\ell) = \sigma_p(S_r) = \sigma_c(S_r) = \emptyset. \end{aligned} \tag{11.7}$$

(iii) For any bounded sequence $\{a_n\}$ in $\mathbb{C}$, let us define the associated *multiplication operator* $L \in \mathcal{L}(\ell^p)$:

$$Lu = (a_1 u_1, ..., a_n u_n, ...) \qquad \forall (u_1, ..., u_n, ...) \in \ell^p, \forall p \in [1, +\infty]. \tag{11.8}$$

It is promptly checked that any a_n is obviously an eigenvalue of L, and the unit vector e_n belongs to the associated eigenspace.

Any cluster point of the sequence $\{a_n\}$ is an element of $\sigma(L)$, as this set is closed. If $a_n \to 0$ then L is compact in ℓ^p for any $p \neq \infty$, as it is a limit of finite-rank operators.

(iv) The point spectrum may be empty. This is the case e.g. for the operator $L : L^2(0,1) \to L^2(0,1)$, with $(Lv)(x) = xv(x)$ for a.e. $x \in]0,1[$. [Ex]

11.3 Spectrum of compact operators

Next we show that, by the Riesz Theorem 10.7,

$$\sigma(K) \setminus \{0\} = \sigma_p(K) \quad \text{(i.e., } [\sigma_c(K) \cup \sigma_r(K)] \subset \{0\}) \qquad \forall K \in \mathcal{K}(X).$$

Here is a more precise statement.

Theorem 11.5 *Let X be a Banach space and $K : X \to X$ be a compact operator. Then:*
(i) the set $\sigma(K)$ is either finite or countably-infinite;
(ii) if $\dim X = \infty$ then $0 \in \sigma(K)$;
(iii) all $\lambda \in \sigma(K) \setminus \{0\}$ are eigenvalues whose eigenspaces have finite dimension;
(iv) 0 is the only possible accumulation point of $\sigma(K)$.

* *Proof.* By the Riesz Theorem 10.7, if $\lambda \neq 0$ then

$$\mathcal{N}(\lambda I - K) = \mathcal{N}(I - K/\lambda) \ \text{ has finite dimension,}$$

and $\mathcal{R}(\lambda I - K) = X$ if $\lambda I - K$ is injective. So $\lambda \notin \sigma_c(K) \cup \sigma_r(K)$, and (iii) is established. If $0 \in \rho(K)$, then K is invertible and $I = K^{-1}K$ is compact, so $\dim X < +\infty$, as it is stated in (ii).

In order to prove (iv), let us assume that there is a sequence $\{\lambda_n\}$ of mutually distinct eigenvalues which converges to some $\lambda \in \mathbb{C}$. Let $\{u_n\}$ be any set of corresponding eigenvectors, and set $M_n = \text{span}\{u_1, \ldots, u_n\}$ for any n. Notice that $K(M_n) \subset M_n$ and $\dim M_n = n$ for any n, since the eigenvectors are linearly independent.

Moreover, for any $v = \sum_{i=1}^n \alpha_i u_i \in M_n$ (with $\alpha_1, ..., \alpha_n \in \mathbb{C}$), we have

$$(\lambda_n I - K)v = \sum_{i=1}^{n-1} \alpha_i (\lambda_n I - K) u_i \in M_{n-1}, \tag{11.9}$$

as $\lambda_n u_n - K u_n = 0$ and $K(M_{n-1}) \subset M_{n-1}$. Thus $(\lambda_n I - K)(M_n) \subset M_{n-1}$.

Because of the Riesz Lemma 2.16, for any $n > 1$ we may choose $v_n \in M_n$ with $\|v_n\| = 1$ and $\text{dist}(v_n, M_{n-1}) \geq 1/2$. For $m < n$, by (11.9) $Kv_m + (\lambda_n v_n - Kv_n) \in M_{n-1} + M_{n-1} = M_{n-1}$. We then have

$$\|Kv_n - Kv_m\| = |\lambda_n| \|v_n - \lambda_n^{-1}(Kv_m + \lambda_n v_n - Kv_n)\| \geq |\lambda_n|/2.$$

Since $\{Kv_n\}$ has a convergent subsequence, we must have $\lambda = 0$. The statement (iv) is thus established, and (i) follows from the boundedness of $\sigma(K)$. □

Examples show that 0 may belong to any part of the spectrum of a compact operator.

11.4 Exercises

1. = Show that the operator $L : L^2(0,1) \to L^2(0,1)$, with $(Lv)(x) = xv(x)$ for a.e. $x \in]0,1[$, has no eigenvector. The point spectrum may thus be empty even for a bounded self-adjoint operator.

A final word

We have seen several things, but much more are missing: the spectral theorem, its extension to unbounded operators, and so on. And afterwards there are applications to partial differential equations, to the calculus of variations, to quantum mechanics, and so on.

So we are at the end of the course, but not of the story.

12 References

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