Distributions and Fourier Transform – A. Visintin – 2014

This chapter includes the following sections:
1. Distributions.
2. Convolution.
3. Fourier transform of functions.
4. Extensions of the Fourier transform.

The symbol [Ex] means that the proof is left as exercise. [] means that a proof is missing.

1 Distributions

The theory of distributions was introduced in the 1940s by Laurent Schwartz, who provided a thorough functional formulation to previous ideas of Heaviside, Dirac and others, and forged a powerful tool of calculus. Distributions also offer a solid basis for the construction of Sobolev spaces, that had been introduced by Sobolev in the 1930s using the notion of *weak derivative.* These spaces play a fundamental role in the modern analysis of linear and nonlinear partial differential equations.

We shall denote by Ω a nonempty domain of $\mathbb{R}^N$. The notion of distribution rests upon the idea of regarding any locally integrable function $f : \Omega \to \mathbb{C}$ as a continuous linear functional acting on a topological vector space $\mathcal{T}(\Omega)$:

$$T_f(v) := \int_\Omega f(x)v(x)\,dx \qquad \forall v \in \mathcal{T}(\Omega). \tag{1.1}$$

One is thus induced to consider all the functionals of the topological dual $\mathcal{T}'(\Omega)$ of $\mathcal{T}(\Omega)$. In this way several classes of distributions are generated. The space $\mathcal{T}(\Omega)$ must be so large that the functional T_f determines a unique f. On the other hand, the smaller is the space $\mathcal{T}(\Omega)$, the larger is its topological dual $\mathcal{T}'(\Omega)$. Moreover, there exists a smallest space $\mathcal{T}(\Omega)$, so that $\mathcal{T}'(\Omega)$ is the largest one; the elements of this dual space are what we name *distributions.*

In this section we outline some basic tenets of this theory, and provide some tools that we will use ahead.

Test Functions. Let Ω be a domain of $\mathbb{R}^N$. By $\boldsymbol{\mathcal{D}(\Omega)}$ we denote the space of infinitely differentiable functions $\Omega \to \mathbb{C}$ whose support is a compact subset of Ω; these are called **test functions.**

The null function is the only analytic function in $\mathcal{D}(\Omega)$, since any element of this space vanishes in some open set. The *bell-shaped* function

$$\rho(x) := \begin{cases} \exp\left[(|x|^2-1)^{-1}\right] & \text{if } |x| < 1, \\ 0 & \text{if } |x| \geq 1 \end{cases} \tag{1.2}$$

belongs to $\mathcal{D}(\mathbb{R}^N)$. By suitably translating ρ and by rescaling w.r.t. x, nontrivial elements of $\mathcal{D}(\Omega)$ are easily constructed for any Ω.

For any $K \subset\subset \Omega$ (i.e., any compact subset K of Ω), let us denote by $\mathcal{D}_K(\Omega)$ the space of the infinitely differentiable functions $\Omega \to \mathbb{C}$ whose support is contained in K. This is a vector subspace of $C^\infty(\Omega)$, and $\mathcal{D}(\Omega) = \bigcup_{K\subset\subset\Omega} \mathcal{D}_K(\Omega)$. The space $\mathcal{D}(\Omega)$ is equipped with the finest topology among those that make all injections $\mathcal{D}_K(\Omega) \to \mathcal{D}(\Omega)$ continuous (so-called *inductive-limit topology).* This topology makes $\mathcal{D}(\Omega)$ a nonmetrizable locally convex Hausdorff space. []

By definition of the inductive-limit topology, a set $A \subset \mathcal{D}(\Omega)$ is open in this topology iff $A \cap \mathcal{D}_K(\Omega)$ is open for any $K \subset\subset \Omega$. Here we shall not study this topology: for our purposes, it will suffice to characterize the corresponding notions of bounded subsets and of convergent sequences.

A subset $B \subset \mathcal{D}(\Omega)$ is bounded in the inductive topology iff it is contained and is bounded in $\mathcal{D}_K(\Omega)$ for some $K \subset\subset \Omega$. [] This means that

(i) there exists a $K \subset\subset \Omega$ that contains the support of all the functions of B, and

(ii) $\sup_{v \in B} \sup_{x \in \Omega} |D^\alpha v(x)| < +\infty$ for any $\alpha \in \mathbb{N}^N$.

As any convergent sequence is necessarily bounded, the following characterization of convergent sequences of $\mathcal{D}(\Omega)$ should be easily understood. A sequence $\{u_n\}$ in $\mathcal{D}(\Omega)$ converges to $u \in \mathcal{D}(\Omega)$ in the inductive topology iff, for some $K \subset\subset \Omega$, $u_n, u \in \mathcal{D}_K(\Omega)$ for any n, and $u_n \to u$ in $\mathcal{D}_K(\Omega)$. [] This means that

(i) there exists a $K \subset\subset \Omega$ that contains the support of any u_n and of u, and

(ii) $\sup_{x \in \Omega} |D^\alpha (u_n - u)(x)| \to 0$ for any $\alpha \in \mathbb{N}^N$. [Ex]

For instance, if ρ is defined as in (1.2), then the sequence $\{\rho(\cdot - a_n)\}$ is bounded in $\mathcal{D}(\mathbb{R})$ iff the sequence $\{a_n\}$ is bounded. Moreover $\rho(\cdot - a_n) \to \rho(\cdot - a)$ in $\mathcal{D}(\mathbb{R}^N)$ iff $a_n \to a$. [Ex]

Distributions. All linear and continuous functionals $\mathcal{D}(\Omega) \to \mathbb{C}$ are called **distributions**; these functionals thus form the (topological) dual space $\boldsymbol{\mathcal{D}'(\Omega)}$. For any $T \in \mathcal{D}'(\Omega)$ and any $v \in \mathcal{D}(\Omega)$ we also write $\langle T, v \rangle$ in place of $T(v)$.

Theorem 1.1 (Characterization of Distributions)
For any linear functional $T : \mathcal{D}(\Omega) \to \mathbb{C}$, the following properties are mutually equivalent:

(i) T is continuous, i.e., $T \in \mathcal{D}'(\Omega)$;

(ii) T is bounded, i.e., it maps bounded subsets of $\mathcal{D}(\Omega)$ to bounded subsets of $\mathbb{C}$;

(iii) T is sequentially continuous, i.e., $T(v_n) \to 0$ whenever $v_n \to 0$ in $\mathcal{D}(\Omega)$;

(iv)

$$\begin{aligned} &\forall K \subset\subset \Omega, \exists m \in \mathbb{N}, \exists C > 0 : \forall v \in \mathcal{D}(\Omega), \\ &\mathrm{supp}(v) \subset K \quad \Rightarrow \quad |T(v)| \leq C \max_{|\alpha| \leq m} \sup_K |D^\alpha v|. \qquad [] \end{aligned} \tag{1.3}$$

(If m is the smallest integer integer such that the latter condition is fulfilled, one says that T has order m on the compact set K; m may actually depend on K.)

Here are some examples of distributions:

(i) For any $f \in L^1_{\mathrm{loc}}(\Omega)$, the integral functional

$$T_f : v \mapsto \int_\Omega f(x)\, v(x)\, dx \tag{1.4}$$

is a distribution. The mapping $f \mapsto T_f$ is injective, so that we may identify $L^1_{\mathrm{loc}}(\Omega)$ with a subspace of $\mathcal{D}'(\Omega)$. These distributions are called *regular;* the others are called *singular.*

(ii) Let μ be either a complex-valued Borel measure on Ω, or a positive measure on Ω that is finite on any $K \subset\subset \Omega$. In either case the functional

$$T_\mu : v \mapsto \int_\Omega v(x)\, d\mu(x) \tag{1.5}$$

is a distribution, that is usually identified with μ itself. in particular this applies to continuous functions.

(iii) Although the function $x \mapsto 1/x$ is not locally integrable in $\mathbb{R}$, its **principal value** (p.v.),

$$\langle p.v. \frac{1}{x}, v \rangle := \lim_{\varepsilon \to 0} \int_{|x| > \varepsilon} \frac{v(x)}{x}\, dx \qquad \forall v \in \mathcal{D}(\mathbb{R}) \tag{1.6}$$

is a distribution. For any $v \in \mathcal{D}(\mathbb{R})$ and for any $a > 0$ such that $\text{supp}(v) \subset [-a, a]$, by the oddness of the function $x \mapsto 1/x$ we have

$$\begin{aligned}\langle p.v.\ \frac{1}{x}, v\rangle &= \lim_{\varepsilon\to 0^+}\left(\int_{\varepsilon<|x|<a}\frac{v(x)-v(0)}{x}\,dx + \int_{\varepsilon<|x|<a}\frac{v(0)}{x}\,dx\right)\\ &= \lim_{\varepsilon\to 0^+}\int_{\varepsilon<|x|<a}\frac{v(x)-v(0)}{x}\,dx = \int_{-a}^{a}\frac{v(x)-v(0)}{x}\,dx.\end{aligned} \tag{1.7}$$

This limit exists and is finite, since by the mean value theorem

$$\left|\int_{\varepsilon<|x|<a}\frac{v(x)-v(0)}{x}\,dx\right| \leq 2a\max_{\mathbb{R}}|v'| \qquad \forall\varepsilon > 0.$$

Notice that the principal value is quite different from other notions of *generalized integral.*

(iv) For any $x_0 \in \Omega$ $(\subset \mathbb{N}^N)$ the **Dirac mass** $\delta_{x_0} : v \mapsto v(x_0)$ is a distribution. [Ex] In particular $\delta_0 \in \mathcal{D}'(\mathbb{R})$.

(v) The series of Dirac masses $\sum_{n=1}^{\infty}\delta_{x_n}/n^2$ is a distribution for any sequence $\{x_n\}$ in Ω. [Ex]

(vi) The series $\sum_{n=1}^{\infty}\delta_{x_n}$ is a distribution iff any $K \subset\subset \Omega$ contains at most a finite number of points of the sequence $\{x_n\}$ (i.e., iff $|x_n| \to +\infty$). [Ex] (Indeed, if this condition is fulfilled, whenever any test function is applied to the series this is reduced to a finite sum.) So for instance

$$\sum_{n=1}^{\infty}\delta_n \in \mathcal{D}'(\mathbb{R}), \quad \sum_{n=1}^{\infty}\delta_{1/n} \in \mathcal{D}'(\mathbb{R}\setminus\{0\}), \quad \text{but} \quad \sum_{n=1}^{\infty}\delta_{1/n} \notin \mathcal{D}'(\mathbb{R}).$$

We equip the space $\mathcal{D}'(\Omega)$ with the sequential (weak star) convergence: for any sequence $\{T_n\}$ and any T in $\mathcal{D}'(\Omega)$,

$$T_n \to T \quad \text{in } \mathcal{D}'(\Omega) \quad \Leftrightarrow \quad T_n(v) \to T(v) \quad \forall v \in \mathcal{D}(\Omega). \tag{1.8}$$

This makes $\mathcal{D}'(\Omega)$ a nonmetrizable locally convex Hausdorff space. []

Proposition 1.2 *If $T_n \to T$ in $\mathcal{D}'(\Omega)$ and $v_n \to v$ in $\mathcal{D}(\Omega)$, then $T_n(v_n) \to T(v)$. []*

Differentiation of Distributions. We define the multiplication of a distribution by a C^∞-function and the differentiation [1] of a distribution via **transposition**:

$$\langle fT, v\rangle := \langle T, fv\rangle \qquad \forall T \in \mathcal{D}'(\Omega), \forall f \in C^\infty(\Omega), \forall v \in \mathcal{D}(\Omega)\,, \tag{1.9}$$

$$\langle \tilde{D}^\alpha T, v\rangle := (-1)^{|\alpha|}\langle T, D^\alpha v\rangle \qquad \forall T \in \mathcal{D}'(\Omega), \forall v \in \mathcal{D}(\Omega), \forall \alpha \in \mathbb{N}^N. \tag{1.10}$$

Via the characterization (1.3), it may be checked that $\tilde{D}^\alpha T$ is a distribution. [Ex] (Actually, by (1.3), the operator $\tilde{D}^\alpha$ may just increase the order of T at most of $|\alpha|$ on any $K \subset\subset \Omega$; see ahead.) Thus any distribution has derivatives of any order. More specifically, for any $f \in C^\infty(\Omega)$, the operators $T \mapsto fT$ and $\tilde{D}^\alpha$ are linear and continuous in $\mathcal{D}'(\Omega)$. [Ex]

The definition (1.9) is consistent with the properties of $L^1_{\text{loc}}(\Omega)$. For any $f \in L^1_{\text{loc}}(\Omega)$, the definition (1.10) is also consistent with partial integration: if $T = T_f$, (1.10) indeed reads

$$\int_\Omega [D^\alpha f(x)]v(x)\,dx = (-1)^{|\alpha|}\int_\Omega f(x)D^\alpha v(x)\,dx \qquad \forall v \in \mathcal{D}(\Omega), \forall \alpha \in \mathbb{N}^N.$$

(No boundary terms appears as the support of v is compact.)

[1] In this section we denote the distributional derivative by $\tilde{D}^\alpha$, and the classical derivative, i.e. the pointwise limit of the difference quotient, by D^α, whenever the latter exists.

By (1.10) and as derivatives commute in $\mathcal{D}(\Omega)$, the same applies to $\mathcal{D}'(\Omega)$, that is,

$$\tilde{D}^\alpha \circ \tilde{D}^\beta T = \tilde{D}^{\alpha+\beta} T = \tilde{D}^\beta \circ \tilde{D}^\alpha T \qquad \forall T \in \mathcal{D}'(\Omega), \forall \alpha, \beta \in \mathbb{N}^N. \tag{1.11}$$

The formula of differentiation of the product is extended as follows:

$$\begin{aligned} &\tilde{D}_i(fT) = (D_i f)T + f\tilde{D}_i T \\ &\forall f \in C^\infty(\Omega), \forall T \in \mathcal{D}'(\Omega), \text{ for } i = 1,..,N; \end{aligned} \tag{1.12}$$

in fact, for any $v \in \mathcal{D}(\Omega)$,

$$\begin{aligned} \langle \tilde{D}_i(fT), v\rangle &= -\langle fT, D_i v\rangle = -\langle T, fD_i v\rangle = \langle T, (D_i f)v\rangle - \langle T, D_i(fv)\rangle \\ &= \langle (D_i f)T, v\rangle + \langle \tilde{D}_i T, fv\rangle = \langle (D_i f)T + f\tilde{D}_i T, v\rangle. \end{aligned}$$

A recursive procedure then yields the extension of the classical Leibniz rule:

$$\begin{aligned} &\tilde{D}^\alpha(fT) = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} (D^{\alpha-\beta} f)\tilde{D}^\beta T \\ &\forall f \in C^\infty(\Omega), \forall T \in \mathcal{D}'(\Omega), \forall \alpha \in \mathbb{N}^N. \quad [Ex] \end{aligned} \tag{1.13}$$

The translation (for $\Omega = \mathbb{R}^N$), the conjugation and other linear operations on functions are also easily extended to distributions via transposition. [Ex]

Comparison with Classical Derivatives.

Theorem 1.3 (Du-Bois Reymond)
For any $f \in C^0(\Omega)$ and any $i \in \{1, \dots, N\}$, the two following conditions are equivalent:
(i) $\widetilde{D}_i f \in C^0(\Omega)$, [2]
(ii) f is classically differentiable w.r.t. x_i at each point of Ω, and $D_i f \in C^0(\Omega)$. []

The next theorem applies to $\Omega :=]a, b[$, for $-\infty \le a < b \le +\infty$. First we remind the reader that

$$\begin{aligned} &\text{a function } f \in L^1(a,b) \text{ is absolutely continuous iff} \\ &\exists g \in L^1(a,b) : \ f(x) = f(y) + \int_y^x g(\xi)\, d\xi \qquad \forall x, y \in]a,b[. \end{aligned}$$

This entails that $f' = g$ a.e. in $]a, b[$. Thus if $f \in L^1(a, b)$ is absolutely continuous, then it is a.e. differentiable (in the classical sense) and $f' \in L^1(a, b)$.

The converse does not hold: even if f is a.e. differentiable and $f' \in L^1(a, b)$, $f \in L^1(a, b)$ need not be absolutely continuous and $\tilde{D}_i f$ need not be a regular distribution. A counterexample is provided by the Heaviside function H:

$$H(x) := 0 \quad \forall x < 0 \qquad H(x) := 1 \quad \forall x \ge 0. \qquad [Ex] \tag{1.14}$$

$DH = 0$ a.e. in $\mathbb{R}$, but of course H is not (a.e. equal to) an absolutely continuous function. Notice that $\tilde{D}H = \delta_0$ since

$$\langle \tilde{D}H, v\rangle = -\int_{\mathbb{R}} H(x)\, Dv(x)\, dx = -\int_{\mathbb{R}^+} Dv(x)\, dx = v(0) = \langle \delta_0, v\rangle \quad \forall v \in \mathcal{D}(\mathbb{R}).$$

[2] that is, $\widetilde{D}_i f$ is a regular distribution that may be identified with a function $h \in C^0(\Omega) \cap L^1_{\rm loc}(\Omega)$. Using the notation (**??**), this condition and the final assertion read $\widetilde{D}_i T_f = T_h$ and $\widetilde{D}_i T_f = T_{D_i f}$ in Ω, respectively.

Theorem 1.4 *A function $f \in L^1(a,b)$ is a.e. equal to an absolutely continuous function $\hat{f}$, iff $\tilde{D}f \in L^1(a,b)$. In this case $\tilde{D}f = D\hat{f}$ a.e. in $]a,b[$.* []

Thus, for complex functions of a single variable:
(i) f is of class C^1 iff f and $\widetilde{D}f$ are both continuous,
(ii) f is absolutely continuous iff f and $\widetilde{D}f$ are both locally integrable.

Henceforth all derivatives will be meant in the sense of distributions, if not otherwise stated. We shall denote them by D^α, dropping the tilde.

Examples. (i) $D\log|x| = 1/x$ (in $\mathbb{R}$) in standard calculus, but not in the theory of distributions, as $1/x$ is not locally integrable in any neighbourhood of $x = 0$, and thus it is no distribution. We claim that, for any $v \in \mathcal{D}(\mathbb{R})$ and any $a > 0$ such that $\mathrm{supp}(v) \subset [-a,a]$,

$$D\log|x| = \text{p.v.}\frac{1}{x} \qquad \text{in } \mathcal{D}'(\mathbf{R}). \tag{1.15}$$

Indeed, as the support of any $v \in \mathcal{D}(\mathbb{R})$ is contained in some symmetric interval $[-a,a]$, we have

$$\begin{aligned}
&\langle D\log|x|, v\rangle = -\langle \log|x|, v'\rangle = -\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}\setminus[-\varepsilon,\varepsilon]} (\log|x|)\, v'(x)\, dx \\
&= \lim_{\varepsilon\to 0^+}\Big\{\int_{[-a,a]\setminus[-\varepsilon,\varepsilon]} \frac{1}{x}\, v(x)\, dx + (\log|\varepsilon|)\,[v(\varepsilon) - v(-\varepsilon)]\Big\} \\
&\Big(\text{as} \int_{[-a,a]\setminus[-\varepsilon,\varepsilon]} \frac{v(0)}{x}\, dx = 0 \;\text{ and }\; \lim_{\varepsilon\to 0^+}(\log|\varepsilon|)\,[v(\varepsilon) - v(-\varepsilon)] = 0\Big) \\
&= \lim_{\varepsilon\to 0^+}\int_{[-a,a]\setminus[-\varepsilon,\varepsilon]} \frac{v(x) - v(0)}{x}\, dx = \langle \text{p.v.}\frac{1}{x}, v\rangle.
\end{aligned} \tag{1.16}$$

(ii) $D[\text{p.v.}(1/x)] \neq -1/x^2$ as the latter is no distribution. Instead, for any $v \in \mathcal{D}(\mathbb{R})$ and any $a > 0$ such that $\mathrm{supp}(v) \subset [-a,a]$, we have

$$\begin{aligned}
&\langle D(\text{p.v.}\frac{1}{x}), v\rangle = -\langle \text{p.v.}\frac{1}{x}, v'\rangle \overset{(1.7)}{=} -\int_{-a}^{a} \frac{v'(x) - v'(0)}{x}\, dx \\
&= -\lim_{a\to+\infty}\int_{-a}^{a} \frac{[v(x) - v(0) - xv'(0)]'}{x}\, dx \\
&= (\text{by partial integration}) \; -\lim_{a\to+\infty}\int_{-a}^{a} \frac{v(x) - v(0) - xv'(0)}{x^2}\, dx.
\end{aligned} \tag{1.17}$$

The latter integral converges, since v has compact support and (by the mean-value theorem) the integrand equals $v''(\xi_x)$, for some ξ_x between 0 and x. (In passing notice that the condition (1.3) is fulfilled.)

* (iii) The even function

$$f(x) = \frac{\sin(1/|x|)}{|x|} \qquad \text{for a.e. } x \in \mathbb{R} \tag{1.18}$$

is not locally (Lebesgue)-integrable in $\mathbb{R}$; hence it cannot be identified with a distribution. On the other hand, it is easily seen that the next two limits exist

$$\begin{aligned}
g(x) &:= \lim_{\varepsilon\to 0+}\int_{\varepsilon}^{x} f(t)\, dt \qquad \forall x > 0, \\
g(x) &:= \lim_{\varepsilon\to 0-}\int_{\varepsilon}^{x} f(t)\, dt \qquad \forall x < 0.
\end{aligned} \tag{1.19}$$

That is, $g(x) := \int_0^x f(t)\,dt$, if this is understood as a generalized Riemann integral. Moreover, $g \in L^1_{\rm loc}(\mathbb{R}) \subset \mathcal{D}'(\mathbb{R})$, so that $Dg \in \mathcal{D}'(\mathbb{R})$; however, Dg cannot be identified with f ($\notin \mathcal{D}'(\mathbb{R})$). Actually, the distribution Dg is a *regularization* of the function f (namely, a distribution T whose restriction to $\mathbb{R} \setminus \{0\}$ coincides with f).

As g is odd and has a finite limit (denoted $g(+\infty)$) at $+\infty$, for any $v \in \mathcal{D}(\mathbb{R})$ and any $a > 0$ such that $\mathrm{supp}(v) \subset [-a, a]$,

$$\begin{aligned} \langle Dg, v\rangle &= -\langle g, v'\rangle = -\lim_{b\to+\infty} \int_{-b}^{b} g(x)[v(x) - v(0)]'\,dx \\ &= \lim_{b\to+\infty} \int_{-b}^{b} f(x)[v(x) - v(0)]\,dx + \lim_{b\to+\infty} [g(b) - g(-b)]v(0) \\ &= \int_{-a}^{a} f(x)[v(x) - v(0)]\,dx + 2g(+\infty)v(0) \qquad \forall v \in \mathcal{D}(\mathbb{R}). \end{aligned} \tag{1.20}$$

* (iv) The modifications for the odd function $\tilde f(x) = [\sin(1/|x|)]/x$ are left to the reader. □

Problems of Division. For any $f \in C^\infty(\mathbb{R}^N)$ and $S \in \mathcal{D}'(\mathbb{R}^N)$, let us consider the problem

$$\text{find } \ T \in \mathcal{D}'(\mathbb{R}^N) \ \text{ such that } \ fT = S. \tag{1.21}$$

(This is named a *problem of division,* since formally $T = S/f$.) The general solution may be represented as the sum of a particular solution of the nonhomogeneous equation and the general solution of the homogeneous equation $fT_0 = 0$. The latter may depend on a number of arbitrary constants.

If f does not vanish in $\mathbb{R}^N$, then $1/f \in C^\infty(\mathbb{R}^N)$ and (1.21) has one and only one solution: $T = (1/f)S$. On the other hand, if f vanish at some points of $\mathbb{R}^N$, the solution is less trivial. Let us see the case of $N = 1$, along the lines of [Gilardi: Analisi 3]. For instance, if $f(x) = x^m$ (with $m \in \mathbb{N}$), then the homogeneous equation $x^m T = 0$ has the general solution $T_0 = \sum_{n=0}^{m-1} c_n D^n \delta_0$, with $c_n \in \mathbb{C}$ for any n. [Ex] On the other hand, even the simple-looking equation $x^m T = 1$ is more demanding: notice that $x^{-m} \notin \mathcal{D}'(\mathbb{R})$ for any integer $m \geq 1$.

Support and Order of Distributions. For any open set $\widetilde\Omega \subset \Omega$ and any $T \in \mathcal{D}'(\Omega)$, we define the restriction of T to $\widetilde\Omega$, denoted $T\big|_{\widetilde\Omega}$, by

$$\langle T\big|_{\widetilde\Omega}, v\rangle := \langle T, v\rangle \qquad \forall v \in \mathcal{D}(\Omega) \text{ such that } \mathrm{supp}(v) \subset \widetilde\Omega.$$

Because of Theorem 1.1, $T\big|_{\widetilde\Omega} \in \mathcal{D}'(\widetilde\Omega)$.

A distribution $T \in \mathcal{D}'(\Omega)$ is said to vanish in an open subset $\tilde\Omega$ of Ω iff it vanishes on any function of $\mathcal{D}(\Omega)$ supported in $\tilde\Omega$. Notice that, for any triplet of Euclidean domains $\Omega_1, \Omega_2, \Omega_3$,

$$\Omega_1 \subset \Omega_2 \subset \Omega_3 \quad \Rightarrow \quad \big(T\big|_{\Omega_2}\big)\big|_{\Omega_1} = T\big|_{\Omega_1} \qquad \forall T \in \mathcal{D}'(\Omega_3). \tag{1.22}$$

There exists then a (possibly empty) largest open set $A \subset \Omega$ in which T vanishes. [Ex] Its complement in Ω is called the **support** of T, and will be denoted by $\mathrm{supp}(T)$.

For any $K \subset\subset \Omega$, the smallest integer m that fulfills the estimate (1.3) is called the order of T in K. The supremum of these orders is called the **order** of T; each distribution is thus of either finite or infinite order. For instance,

(i) regular distributions and the Dirac mass are of order zero; [Ex]

(ii) $D^\alpha \delta_0$ is of order $|\alpha|$ for any $\alpha \in \mathbb{N}^N$;

(iii) p.v. $(1/x)$ is of order one in $\mathcal{D}'(\mathbb{R})$. [Ex]

On the other hand, $\sum_{n=1}^\infty D^n \delta_n$ is of infinite order in $\mathcal{D}'(\mathbb{R})$.

The next statement directly follows from (1.3).

Theorem 1.5 *Any compactly supported distribution is of finite order.*

The next theorem is also relevant, and will be applied ahead.

Theorem 1.6 *Any distribution whose support is the origin is a finite combination of derivatives of the Dirac mass.*

The Space $\mathcal{E}(\Omega)$ and its Dual. In his theory of distributions, Laurent Schwartz denoted by $\mathcal{E}(\Omega)$ the space $C^\infty(\Omega)$, equipped with the family of seminorms

$$|v|_{K,\alpha} := \sup_{x\in K} |D^\alpha v(x)| \qquad \forall K \subset\subset \Omega, \forall \alpha \in \mathbb{N}^N.$$

This renders $\mathcal{E}(\Omega)$ a locally convex Frèchet space, and induces the topology of uniform convergence of all derivatives on any compact subset of Ω: for any sequence $\{u_n\}$ in $\mathcal{E}(\Omega)$ and any $u \in \mathcal{E}$,

$$\begin{aligned} &u_n \to u \quad \text{in } \mathcal{E}(\Omega) \quad \Leftrightarrow \\ &\sup_{x\in K} |D^\alpha(u_n - u)(x)| \to 0 \quad \forall K \subset\subset \Omega, \quad \forall \alpha \in \mathbb{N}^N. \end{aligned} \tag{1.23}$$

Notice that

$$\mathcal{D}(\Omega) \subset \mathcal{E}(\Omega) \quad \text{with continuous and sequentially dense injection,} \tag{1.24}$$

namely, any element of $\mathcal{E}(\Omega)$ may be approximated by a sequence of $\mathcal{D}(\Omega)$. This may be checked via multiplication by a suitable sequence of compactly supported smooth functions. [Ex] By (1.24)

$$\mathcal{E}'(\Omega) \subset \mathcal{D}'(\Omega) \quad \text{with continuous and sequentially dense injection,} \tag{1.25}$$

so that we may identify $\mathcal{E}'(\Omega)$ with a subspace of $\mathcal{D}'(\Omega)$.

As we did for $\mathcal{D}'(\Omega)$, we shall equip the space $\mathcal{E}'(\Omega)$ with the sequential weak star convergence: for any sequence $\{T_n\}$ in $\mathcal{E}'(\Omega)$ and any $T \in \mathcal{E}'(\Omega)$,

$$T_n \to T \quad \text{in } \mathcal{E}'(\Omega) \quad \Leftrightarrow \quad T_n(v) \to T(v) \quad \forall v \in \mathcal{E}(\Omega). \tag{1.26}$$

[This makes $\mathcal{E}'(\Omega)$ a nonmetrizable locally convex Hausdorff space.]

The sequential weak star convergence of $\mathcal{E}'(\Omega)$ is strictly stronger than that induced by $\mathcal{D}'(\Omega)$: for any sequence $\{T_n\}$ in $\mathcal{E}'(\Omega)$ and any $T \in \mathcal{E}'(\Omega)$,

$$T_n \to T \quad \text{in } \mathcal{E}'(\Omega) \quad \overset{\not\Leftarrow}{\Rightarrow} \quad T_n \to T \quad \text{in } \mathcal{D}'(\Omega).[Ex] \tag{1.27}$$

If $\Omega = \mathbb{R}$, the sequence $\{\chi_{[n,n+1]}\}$ (the characteristic functions of the intervals $[n, n+1]$) is a counterexample to the converse implication:

$$\chi_{[n,n+1]} \to 0 \qquad \text{in } \mathcal{D}'(\mathbb{R}^N) \text{ but not in } \mathcal{E}'(\mathbb{R}^N).$$

Theorem 1.7 $\mathcal{E}'(\Omega)$ *may be identified with the subspace of distributions having compact support.*

We just outline a part of the argument. Let $T \in \mathcal{D}'(\Omega)$ have support $K \subset\subset \Omega$. For any $v \in \mathcal{E}(\Omega)$, multiplying it by χ_K and then convoluting with a regularizing kernel ρ (see (1.2)), one may construct $v_0 \in \mathcal{D}(\Omega)$ such that $v_0 = v$ in K. [Ex] One may thus define $\tilde{T}(v)$ by setting $\tilde{T}(v) = T(v_0)$. It is easily checked that this determines a unique $\tilde{T} \in \mathcal{E}'(\Omega)$. Compactly supported distributions may thus be identified with certain elements of $\mathcal{E}'(\Omega)$.

The proof of the surjectivity of the mapping $T \mapsto \tilde{T}$ is less straightforward, and is here omitted.

On the basis of the latter theorem, examples of elements of $\mathcal{E}'(\Omega)$ are easily provided. E.g.:
(i) any compactly supported $f \in L^1_{\rm loc}$ belongs to $\mathcal{E}'(\Omega)$,
(ii) $\sum_{n=1}^m D^{\alpha_n}\delta_{a_n} \in \mathcal{E}'(\Omega)$, for any finite families $a_1, ..., a_m \in \Omega$ and $\alpha_1, ..., \alpha_m \in \mathbb{N}^N$,
(iii) $\sum_{n=1}^\infty n^{-2} D^{\alpha_n}\delta_{a_n} \in \mathcal{E}'(\Omega)$, for any sequence $\{a_n\}$ contained in a compact subset of Ω, and any sequence of multi-indices $\{\alpha_n\}$. (If the coefficients n^{-2} are dropped or the sequence $\{a_n\}$ is not confined to a compact subset of Ω, this is no element of $\mathcal{E}'(\Omega)$.)

On the basis of the latter theorem, we may apply to $\mathcal{E}'(\Omega)$ the operations that we defined for distributions. It is straightforward to check that this space is stable by differentiation, multiplication by a smooth function, and so on.

The Space $\mathcal{S}$ of Rapidly Decreasing Functions. In order to extend the Fourier transform to distributions, Laurent Schwartz introduced the space of **(infinitely differentiable) rapidly decreasing functions** (at ∞): [3]

$$\begin{aligned} \mathcal{S}\big(\mathbb{R}^N\big) :=& \big\{v \in C^\infty : \forall \alpha, \beta \in \mathbb{N}^N, x^\beta D^\alpha v \in L^\infty\big\} \\ =& \big\{v \in C^\infty : \forall \alpha \in \mathbb{N}^N, \forall m \in \mathbb{N}, \\ & |x|^m D^\alpha v(x) \to 0 \text{ as } |x| \to +\infty\big\}. \end{aligned} \tag{1.28}$$

(The latter equality is easily checked.) [Ex] We shall write $\mathcal{S}$ in place of $\mathcal{S}\big(\mathbb{R}^N\big)$. This is a locally convex Fréchet space equipped with either of the following equivalent families of seminorms []

$$|v|_{\alpha,\beta} := \sup_{x \in \mathbb{R}^N} |x^\beta D^\alpha v(x)| \qquad \alpha, \beta \in \mathbb{N}^N, \tag{1.29}$$

$$|v|_{m,\alpha} := \sup_{x \in \mathbb{R}^N} (1 + |x|^2)^m |D^\alpha v(x)| \qquad m \in \mathbb{N}, \alpha \in \mathbb{N}^N. \tag{1.30}$$

For instance, for any $\theta \in C^\infty$ such that $\theta(x)/|x|^a \to +\infty$ as $|x| \to +\infty$ for some $a > 0$, $e^{-\theta(x)} \in \mathcal{S}$. By the Leibniz rule, for any polynomials P and Q, the operators

$$u \mapsto P(x)Q(D)u, \qquad u \mapsto P(D)[Q(x)u] \tag{1.31}$$

map $\mathcal{S}$ to $\mathcal{S}$ and are continuous. [Ex] It is easily checked that

$$\mathcal{D} \subset \mathcal{S} \subset \mathcal{E} \quad \text{with continuous and sequentially dense injections.} \tag{1.32}$$

The Space $\mathcal{S}'$ of Tempered Distributions. We shall denote the (topological) dual space of $\mathcal{S}$ by $\mathcal{S}'$. As $\mathcal{S}$ is a metric space, this is the space of the linear functionals $T : \mathcal{S} \to \mathbb{C}$ such that

$$\{v_n\} \subset \mathcal{S}, \quad v_n \to 0 \quad \text{in } \mathcal{S} \quad \Rightarrow \quad \langle T, v_n\rangle \to 0. \tag{1.33}$$

The elements of this space are usually named **tempered distributions:** we shall see that actually $\mathcal{S}' \subset \mathcal{D}'$ (up to identifications) with continuous injection. Here are some examples:
(i) any compactly supported $T \in \mathcal{D}'(\Omega)$,

[3] Laurent Schwartz founded the theory of distributions upon the dual of three main function spaces: $\mathcal{D}(\Omega)$, $\mathcal{E}(\Omega)$ and $\mathcal{S}(\mathbb{R}^N)$. The two latter are Fréchet space, at variance with the first one; the same holds for the respective (topological) duals.
Notice that this does not subsume any monotonicity property; e.g., the nonmonotone function $e^{-|x|^2} \sin x$ is an element of $\mathcal{S}(\mathbb{R})$.

(ii) any $f \in L^p$ with $p \in [1, +\infty]$,
(iii) any function f such that $|f(x)| \le C(1 + |x|)^m$ for some $C > 0$ and $m \in \mathbb{N}$,
(iv) $f(x) = p(x)w(x)$, for any polynomial p and any $w \in L^1$. [Ex]

On the other hand $L^1_{\rm loc}$ is not included in $\mathcal{S}'$. E.g., $e^{|x|} \notin \mathcal{S}'$. Nevertheless ahead we shall see that $e^x \cos(e^x) \in \mathcal{S}'$ for $N = 1$, at variance with what might be expected.

Convergence in $\mathcal{S}'$. As we did for $\mathcal{D}'(\Omega)$ and $\mathcal{E}'(\Omega)$, we shall equip the space $\mathcal{S}'$ with the sequential weak star convergence: for any sequence $\{T_n\}$ in $\mathcal{S}'$ and any $T \in \mathcal{S}'$,

$$T_n \to T \quad \text{in } \mathcal{S}' \quad \Leftrightarrow \quad T_n(v) \to T(v) \quad \forall v \in \mathcal{S}. \tag{1.34}$$

[This makes $\mathcal{S}'$ a nonmetrizable locally convex Hausdorff space.]

As $\mathcal{D} \subset \mathcal{S}' \subset \mathcal{D}'$ and $\mathcal{D}$ is a sequentially dense subset of $\mathcal{D}'$, it follows that

$$\mathcal{S}' \subset \mathcal{D}' \quad \text{with continuous and sequentially dense injection; } [Ex] \tag{1.35}$$

namely, any element of $\mathcal{D}'$ may be approximated by a sequence of $\mathcal{S}'$. The sequential weak star convergence of $\mathcal{S}'$ is strictly stronger than that induced by $\mathcal{D}'$: for any sequence $\{T_n\}$ in $\mathcal{S}'$ and any $T \in \mathcal{S}'$,

$$T_n \to T \quad \text{in } \mathcal{S}' \quad \overset{\not\Leftarrow}{\Rightarrow} \quad T_n \to T \quad \text{in } \mathcal{D}'.[Ex] \tag{1.36}$$

In $\mathbb{R}$, $\{e^{|x|}\chi_{[n,n+1]}\}$ is a counterexample to the converse implication:

$$e^{|x|}\chi_{[n,n+1]} \to 0 \qquad \text{in } \mathcal{D}' \text{ but not in } \mathcal{S}'. \tag{1.37}$$

On the other hand $L^1_{\rm loc}$ is not included in $\mathcal{S}'$, not even for $N = 1$. E.g., $e^{|x|} \notin \mathcal{S}'$.

Because of (1.35), we may apply to $\mathcal{S}'$ the operations that we defined for distributions. It is straightforward to check that this space is stable by differentiation, multiplication by a smooth function, and so on.

Overview of Distribution Spaces. We introduced the spaces $\mathcal{D}(\Omega), \mathcal{E}(\Omega)$, with (up to identifications)

$$\mathcal{D}(\Omega) \subset \mathcal{E}(\Omega) \quad \text{with continuous and dense injection.} \tag{1.38}$$

For $\Omega = \mathbb{R}^N$ (which is not displayed), we also defined $\mathcal{S}$, which is such that

$$\mathcal{D} \subset \mathcal{S} \subset \mathcal{E} \quad \text{with continuous and dense injection.} \tag{1.39}$$

We equipped the respective dual spaces with the weak star convergence. (1.38) and (1.39) respectively yield

$$\mathcal{E}'(\Omega) \subset \mathcal{D}'(\Omega) \quad \text{with continuous and sequentially dense injection,} \tag{1.40}$$

and, for $\Omega = \mathbb{R}^N$,

$$\mathcal{E}' \subset \mathcal{S}' \subset \mathcal{D}' \quad \text{with continuous and sequentially dense injection.} \tag{1.41}$$

(The density of the inclusions $\mathcal{S} \subset \mathcal{E}$ and $\mathcal{E}' \subset \mathcal{S}'$ was not mentioned above. However the former one is obvious, and the second one directly follows from it.)

L. Schwartz also introduced spaces of slowly increasing functions and rapidly decreasing distributions. But we shall not delve on them.

Exercises.

2 Convolution

Convolution of L^1-Functions. For any measurable functions $f, g : \mathbb{R}^N \to \mathbb{C}$, we call **convolution product** (or just **convolution)** of f and g the function

$$(f * g)(x) := \int f(x-y)g(y)\, dy \qquad \text{for a.e. } x \in \mathbb{R}^N, \tag{2.42}$$

whenever this integral converges (absolutely) for a.e. x. (We write $\int ...dy$ in place of $\int ... \int_{\mathbb{R}^N} ... dy_1 ... dy_N$, and omit to display the domain $\mathbb{R}^N$.) Note that

$$\operatorname{supp}(f * g) \subset \operatorname{supp}(f) + \operatorname{supp}(g). \quad [Ex] \tag{2.43}$$

If A and B are two topological vector spaces of functions for which the convolution makes sense, we set $A * B := \{f * g : f \in A, g \in B\}$, and define $A \cdot B$ similarly.

Proposition 2.1 *(i) $L^1 * L^1 \subset L^1$, and*

$$\|f * g\|_{L^1} \le \|f\|_{L^1} \|g\|_{L^1} \qquad \forall f, g \in L^1 . \tag{2.44}$$

(ii) $L^1_{\text{loc}} * L^1_{\text{comp}} \subset L^1_{\text{loc}}$, and [4]

$$\begin{aligned} &\|f * g\|_{L^1(K)} \le \|f\|_{L^1(K - \operatorname{supp}(g))} \|g\|_{L^1} \\ &\forall K \subset\subset \mathbb{R}^N, \forall f \in L^1_{\text{loc}}, \forall g \in L^1_{\text{comp}}. \end{aligned} \tag{2.45}$$

Moreover $L^1_{\text{comp}} * L^1_{\text{comp}} \subset L^1_{\text{comp}}$.

(iii) For $N = 1$, $L^1_{\text{loc}}(\mathbb{R}^+) * L^1_{\text{loc}}(\mathbb{R}^+) \subset L^1_{\text{loc}}(\mathbb{R}^+)$. [5] *For any $f, g \in L^1_{\text{loc}}(\mathbb{R}^+)$,*

$$(f * g)(x) = \begin{cases} \int_0^x f(x-y)g(y)\, dy & \text{for a.e. } x \ge 0 \\ 0 & \text{for a.e. } x < 0, \end{cases} \tag{2.46}$$

$$\|f * g\|_{L^1(0,M)} \le \|f\|_{L^1(0,M)} \|g\|_{L^1(0,M)} \qquad \forall M > 0. \tag{2.47}$$

The mapping $(f, g) \mapsto f * g$ is thus continuous in each of these three cases.

Proof. (i) For any $f, g \in L^1$, the function $(\mathbb{R}^N)^2 \to \mathbb{C} : (z, y) \mapsto f(z)g(y)$ is (absolutely) integrable, and by changing integration variable we get

$$\iint f(z)g(y)\, dz\, dy = \iint f(x-y)g(y)\, dy\, dx.$$

By Fubini's theorem the function $f * g : x \mapsto \int f(x-y)g(y)\, dy$ is then integrable. Moreover

$$\begin{aligned} \|f * g\|_{L^1} &= \int dx \left| \int f(x-y)g(y)\, dy \right| \\ &\le \iint |f(x-y)||g(y)|\, dx\, dy = \iint |f(z)||g(y)|\, dz\, dy = \|f\|_{L^1} \|g\|_{L^1}. \end{aligned}$$

[4] By L^1_{comp} we denote the space of integral functions that have compact support.

[5] Any function or distribution defined on $\mathbb{R}^+$ will be automatically extended to the whole $\mathbb{R}$ with value 0. (In signal theory, the functions of time that vanish for any $t < 0$ are said *causal).*

(ii) For any $f \in L^1_{\rm loc}$ and $g \in L^1_{\rm comp}$, setting $S_g := \mathrm{supp}(g)$,

$$(f * g)(x) = \int_{S_g} f(x-y)g(y)\,dy \qquad \text{for a.e. } x \in \mathbb{R}^N.$$

Moreover, for any $K \subset\subset \mathbb{R}^N$,

$$\begin{aligned}\|f * g\|_{L^1(K)} &\le \int_K dx \int_{S_g} |f(x-y)g(y)|\,dy = \int_{S_g} dy \int_K |f(x-y)g(y)|\,dx \\ &= \int_{S_g} dy \int_{K-S_g} |f(z)g(y)|\,dz \le \|f\|_{L^1(K-S_g)}\|g\|_{L^1}.\end{aligned}$$

The proof of the inclusion $L^1_{\rm comp} * L^1_{\rm comp} \subset L^1_{\rm comp}$ is based on (2.43), and is left to the Reader.

(iii) Part (iii) may be proved by means of an argument similar to that of part (ii), that we also leave to the reader. □

Proposition 2.2 *L^1, $L^1_{\rm comp}$ and $L^1_{\rm loc}(\mathbb{R}^+)$, equipped with the convolution product, are commutative algebras (without unit).* [6] *In particular,*

$$\begin{aligned}&f * g = g * f, \qquad (f*g)*h = f*(g*h) \qquad \text{a.e. in } \mathbb{R}^N \\ &\forall (f,g,h) \in (L^1)^3 \cup \big(L^1_{\rm loc} \times L^1_{\rm comp} \times L^1_{\rm comp}\big).\end{aligned} \tag{2.48}$$

If $N = 1$, the same holds for any $(f,g,h) \in L^1_{\rm loc}(\mathbb{R}^+)^3$, too.

Proof. For any $(f,g,h) \in (L^1)^3$ and a.e. $x \in \mathbb{R}^N$,

$$\begin{aligned}(f*g)(x) &= \int f(x-y)g(y)\,dy = \int f(z)g(x-z)dz = (g*f)(x), \\ [(f*g)*h](x) &= \int [(f*g)](z)\,h(x-z)\,dz = \int dz \int f(y)g(z-y)\,dy\ h(x-z) \\ &= \iint f(y)g(t)h((x-y)-t)\,dt\,dy \\ &= \int dy\, f(y) \int g(t)h(x-y-t)\,dt \\ &= \int f(y)[(g*h)](x-y)\,dy = [f*(g*h)](x).\end{aligned}$$

The cases of $(f,g,h) \in \big(L^1_{\rm loc} \times L^1_{\rm comp} \times L^1_{\rm comp}\big)$ and $(f,g,h) \in L^1_{\rm loc}(\mathbb{R}^+)^3$ are similarly checked. □

It is easily seen that $(L^1, *)$ and $(L^\infty, \cdot)$ (here "$\cdot$" stands for the pointwise product) are commutative Banach algebras; $(L^\infty, \cdot)$ has the unit element $e \equiv 1$.

[6] Let a vector space X over a field $\mathbb{K}$ ($= \mathbb{C}$ or $\mathbb{R}$) be equipped with a product $* : X \times X \to X$. This is called an **algebra** iff, for any $u, v, z \in X$ and any $\lambda \in \mathbb{K}$:

(i) $u * (v * z) = (u * v) * z$,

(ii) $(u + v) * z = u * z + v * z, \quad z * (u + v) = z * u + z * v$,

(iii) $\lambda(u * v) = (\lambda u) * v = u * (\lambda v)$.

The algebra is said **commutative** iff the product $*$ is commutative.

X is called a **Banach algebra** iff it is both an algebra and a Banach space (over the same field), and, denoting the norm by $\|\cdot\|$,

(iv) $\|u * v\| \le \|u\|\|v\|$ for any $u, v \in X$.

X is called a **Banach algebra with unit** iff

(v) there exists (a necessarily unique) $e \in X$ such that $\|e\| = 1$, and $e * u = u * e = u$ for any $u \in X$.

(If the unit is missing, it may be constructed in a canonical way...)

Convolution of L^p-Functions. The following result generalizes Proposition 2.1. [7]

• **Theorem 2.3 (Young)** *Let*

$$p, q, r \in [1, +\infty], \qquad p^{-1} + q^{-1} = 1 + r^{-1}.^{8} \tag{2.50}$$

Then: (i) $L^p * L^q \subset L^r$ *and*

$$\|f * g\|_{L^r} \le \|f\|_{L^p}\|g\|_{L^q} \qquad \forall f \in L^p, \forall g \in L^q. \tag{2.51}$$

(ii) $L^p_{\rm loc} * L^q_{\rm comp} \subset L^r_{\rm loc}$ *and*

$$\begin{aligned} &\|f * g\|_{L^r(K)} \le \|f\|_{L^p(K-{\rm supp}(g))}\|g\|_{L^q} \\ &\forall K \subset\subset \mathbb{R}^N, \forall f \in L^p_{\rm loc}, \forall g \in L^q_{\rm comp}. \end{aligned} \tag{2.52}$$

Moreover $L^p_{\rm comp} * L^q_{\rm comp} \subset L^r_{\rm comp}$.

(iii) For $N = 1$, $L^p_{\rm loc}(\mathbb{R}^+) * L^q_{\rm loc}(\mathbb{R}^+) \subset L^r_{\rm loc}(\mathbb{R}^+)$, *and*

$$\begin{aligned} &\|f * g\|_{L^r(0,M)} \le \|f\|_{L^p(0,M)}\|g\|_{L^q(0,M)} \\ &\forall M > 0, \forall f \in L^p_{\rm loc}(\mathbb{R}^+), \forall g \in L^q_{\rm loc}(\mathbb{R}^+). \end{aligned} \tag{2.53}$$

The mapping $(f, g) \mapsto f * g$ is thus continuous in each of these three cases.

Proof. (i) If $p = +\infty$, then by (2.8) $q = 1$ and $r = +\infty$, and (2.51) obviously holds; let us then assume that $p < +\infty$. For any fixed $f \in L^p$, the generalized (integral) Minkowski inequality and the Hölder inequality respectively yield

$$\|f * g\|_{L^p} = \Big\| \int f(x-y)g(y)\,dy \Big\|_{L^p} \le \|f\|_{L^p}\|g\|_{L^1} \qquad \forall g \in L^1,$$

$$\|f * g\|_{L^\infty} = {\rm ess\,sup} \int f(x-y)g(y)\,dy \le \|f\|_{L^p}\|g\|_{L^{p'}} \qquad \forall g \in L^{p'}$$

$(p^{-1} + (p')^{-1} = 1)$. Thus the mapping $g \mapsto f * g$ is (linear and) continuous from L^1 to L^p and from $L^{p'}$ to L^∞. By the Riesz-Thorin Theorem (see below), this mapping is then continuous from L^q to L^r and inequality (2.51) holds, provided that

$$\exists \theta \in \,]0,1[: \quad \frac{1}{q} = \frac{\theta}{1} + \frac{1-\theta}{p'}, \quad \frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{\infty}.$$

As the latter equality yields $\theta = p/r$, by the first one we get $p^{-1} + q^{-1} = 1 + r^{-1}$.

(ii) For any $f \in L^p_{\rm loc}$ and $g \in L^q_{\rm comp}$, setting $S_g := {\rm supp}(g)$,

$$(f * g)(x) = \int_{S_g} f(x-y)g(y)\,dy \quad \text{converges for a.e. } x \in \mathbb{R}^N.$$

---

[7] This theorem may be compared with the following result, that easily follows from the Hölder inequality: If $p, q, r \in [1, +\infty[$ are such that $p^{-1} + q^{-1} = r^{-1}$, then

$$uv \in L^r(\Omega), \quad \|uv\|_r \le \|u\|_p\|v\|_q \qquad \forall u \in L^p(\Omega), \forall v \in L^q(\Omega).[Ex] \tag{2.49}$$

[8] Here we set $(+\infty)^{-1} := 0$.

If $r = +\infty$ then $p = q = 1$, and we are in the situation of part (ii) of Proposition 2.1; let us then assume that $r \neq +\infty$. For any $K \subset\subset \mathbb{R}^N$, denoting by $\chi_{K,g}$ the characteristic function of $K - S_g$, we have

$$\begin{aligned}\|f * g\|^r_{L^r(K)} &= \int_K \left| \int_{S_g} f(x-y)g(y)\,dy \right|^r dx \\ &\leq \int \left| \int (\chi_{K,g} f)(x-y)g(y)\,dy \right|^r dx.\end{aligned}$$

As $\chi_{K,g} f \in L^p$, by part (i) the latter integral is finite.

(iii) Part (iii) may be proved by means of an argument similar to that of part (ii), that we leave to the reader. □

Theorem 2.4 (Riesz-Thorin) * *Let Ω, Ω' be nonempty open subsets of $\mathbb{R}^N$. For $i = 1, 2$, let $p_i, q_i \in [1, +\infty]$ and assume that*

$$T : L^{p_1}(\Omega) + L^{p_2}(\Omega) \to L^{q_1}(\Omega') + L^{q_2}(\Omega') \tag{2.54}$$

is a linear operator such that

$$T : L^{p_i}(\Omega) \to L^{q_i}(\Omega') \text{ is continuous.} \tag{2.55}$$

Let $\theta \in]0, 1[$, and $p := p(\theta)$, $q := q(\theta)$ be such that

$$\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}, \qquad \frac{1}{q} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2}. \tag{2.56}$$

Then T maps $L^p(\Omega)$ to $L^q(\Omega')$, is linear and continuous. Moreover, if M_1 and M_2 are two constants such that

$$\|Tf\|_{L^{q_i}(\Omega')} \leq M_i \|f\|_{L^{p_i}(\Omega)} \qquad \forall f \in L^{p_i}(\Omega) \ (i = 1, 2), \tag{2.57}$$

then

$$\|Tf\|_{L^q(\Omega')} \leq M_1^\theta M_2^{1-\theta} \|f\|_{L^p(\Omega)} \qquad \forall f \in L^p(\Omega). \qquad [] \tag{2.58}$$

By this result, we may regard $L^{p(\theta)}(\Omega)$ as an *interpolate space* between $L^{p_1}(\Omega)$ and $L^{p_2}(\Omega)$. ((2.58) is accordingly called the *interpolate inequality.)* This theorem is actually a prototype of the theory of Banach spaces interpolation.

For any $f : \mathbb{R}^N \to \mathbb{C}$, let us set $\check{f}(x) = f(-x)$.

Corollary 2.5 *Let*

$$p, q, s \in [1, +\infty], \qquad p^{-1} + q^{-1} + s^{-1} = 2. \tag{2.59}$$

Then:

$$\begin{aligned}&\forall (f, g, h) \in L^p \times L^q \times L^s, \\ &(f * g) \cdot h, \ g \cdot (\check{f} * h), \ f \cdot (\check{g} * h) \in L^1, \quad \textit{and} \\ &\int (f * g) \cdot h = \int g \cdot (\check{f} * h) = \int f \cdot (\check{g} * h).\end{aligned} \tag{2.60}$$

The same holds also

$$\begin{aligned}&\forall (f, g, h) \in \left(L^p_{\text{comp}} \times L^q_{\text{loc}} \times L^s_{\text{comp}}\right), \\ &\forall (f, g, h) \in L^p_{\text{loc}}(\mathbb{R}^+) \times L^q_{\text{loc}}(\mathbb{R}^+) \times L^s_{\text{comp}}(\mathbb{R}^+).\end{aligned} \tag{2.61}$$

(In the language of operator theory, $\check{f}$ is the adjoint of the operator $f*$.)

Proof. For any $(f,g,h) \in L^p \times L^q \times L^s$, by the Young Theorem 2.50 $f * g \in L^r$ for r as in (2.50). By (2.59) then $r^{-1} + s^{-1} = 1$, and (2.60) follows.

The remainder is similarly checked. □

Let us next set $\tau_h f(x) := f(x+h)$ for any $f : \mathbb{R}^N \to \mathbb{C}$ and any $x, h \in \mathbb{R}^N$.

Let us denote by $C^0(\mathbb{R}^N)$ the space of continuous functions $\mathbb{R}^N \to \mathbb{C}$ (which is a Fréchet space equipped with the family of sup-norms on compact subsets of $\mathbb{R}^N$), and by $C^0_0(\mathbb{R}^N)$ the subspace of $C^0(\mathbb{R}^N)$ of functions that vanish at infinity (this is a Banach space equipped with the sup-norm).

Lemma 2.6 *As $h \to 0$,*

$$\tau_h f \to f \qquad \text{in } C^0,\ \forall f \in C^0, \tag{2.62}$$

$$\tau_h f \to f \qquad \text{in } L^p,\ \forall f \in L^p,\ \forall p \in [1, +\infty[. \tag{2.63}$$

Proof. As any $f \in C^0$ is locally uniformly continuous, $\tau_h f \to f$ uniformly in any $K \subset\subset \mathbb{R}^N$; (2.62) thus holds. This yields (2.63), as C^0 is dense in L^p for any $p \in [1, +\infty[$. □

By the next result, in the Young theorem the space L^∞ may be replaced by $L^\infty \cap C^0$, and in part (i) also by $L^\infty \cap C^0_0$.

Proposition 2.7 *Let $p, q \in [1, +\infty]$ be such that $p^{-1} + q^{-1} = 1$. Then:*

$$f * g \in C^0 \qquad \forall (f,g) \in (L^p \times L^q) \cup (L^p_{\text{loc}} \times L^q_{\text{comp}}), \tag{2.64}$$

$$f * g \in C^0 \qquad \forall (f,g) \in L^p_{\text{loc}}(\mathbb{R}^+) \times L^q_{\text{loc}}(\mathbb{R}^+) \quad \text{if } N = 1, \tag{2.65}$$

$$(f * g)(x) \to 0 \quad \text{as} \quad |x| \to +\infty \quad \forall (f,g) \in L^p \times L^q, \forall p, q \in [1, +\infty[. \tag{2.66}$$

Proof. For instance, let $p \neq +\infty$ and $(f,g) \in L^p \times L^q$; the other cases may be dealt with similarly. By Lemma 2.6,

$$\begin{aligned} \|\tau_h(f * g) - (f * g)\|_{L^\infty} &= \Big\| \int [f(x+h-y) - f(x-y)] g(y) \, dy \Big\|_{L^\infty} \\ &\leq \|\tau_h f - f\|_{L^p} \|g\|_{L^q} \to 0 \qquad \text{as } h \to 0; \end{aligned} \tag{2.67}$$

the function $f * g$ may then be identified with a uniformly continuous function.

Let $\{f_n\} \subset L^p_{\text{comp}}$ and $\{g_n\} \subset L^q_{\text{comp}}$ be such that $f_n \to f$ in L^p and $g_n \to g$ in L^q. Hence $f_n * g_n$ has compact support, and $f_n * g_n \to f * g$ uniformly. This yields the final statement of the theorem. □

It is easily seen that if either p or $q = +\infty$ then (2.66) fails.

Convolution of Distributions. By part (ii) of Proposition 2.1,

$$f * g \in L^1_{\text{loc}} \qquad \forall (f,g) \in (L^1_{\text{loc}} \times L^1_{\text{comp}}) \cup (L^1_{\text{comp}} \times L^1_{\text{loc}}).$$

For any $\varphi \in \mathcal{D}$, then

$$\int (f * g)(x) \varphi(x) \, dx = \iint f(x-y) g(y) \varphi(x) \, dx dy = \iint f(z) g(y) \varphi(z+y) \, dz dy, \tag{2.68}$$

and of course each of these double integrals equals the corresponding iterated integrals, by Fubini's theorem. This formula allows one to extend the operation of convolution to distributions, under analogous restrictions on the supports. Let either $(T,S) \in (\mathcal{D}' \times \mathcal{E}') \cup (\mathcal{E}' \times \mathcal{D}')$, and define

$$\langle T * S, \varphi \rangle := \langle T_x, \langle S_y, \varphi(x+y) \rangle \rangle \qquad \forall \varphi \in \mathcal{D}. \tag{2.69}$$

(In $\langle S_y, \varphi(x+y)\rangle$ the variable x is just a parameter; if this pairing is reduced to an integration, then y is the integration variable.) This is meaningful, since

$$S \in \mathcal{E}' \ \ (S \in \mathcal{D}', \text{ resp.}) \quad \Rightarrow \quad \langle S_y, \varphi(x+y)\rangle \in \mathcal{D} \ \ (\in \mathcal{E}, \text{ resp.}). \ [Ex] \tag{2.70}$$

For $N = 1$, if $T \in \mathcal{D}'(\mathbb{R}^+)$, then (2.69) still makes sense.

On the other hand, one cannot write $\langle T_x S_y, \varphi(x+y)\rangle$ in the duality between $\mathcal{D}'(\mathbb{R}^N \times \mathbb{R}^N)$ and $\mathcal{D}(\mathbb{R}^N \times \mathbb{R}^N)$, since the support of the mapping $(x,y) \mapsto \varphi(x+y)$ is compact only if $\varphi \equiv 0$.

In $\mathcal{E}'$ the convolution commutes and is associative. Thus $(\mathcal{E}', *)$ is a convolution algebra, with unit element δ_0. Here are some further properties:

$$\mathcal{D}' * \mathcal{E}' \subset \mathcal{D}', \qquad \mathcal{E}' * \mathcal{E}' \subset \mathcal{E}', \tag{2.71}$$

$$\mathcal{S}' * \mathcal{E}' \subset \mathcal{S}', \qquad \mathcal{S} * \mathcal{S}' \subset \mathcal{E} \cap \mathcal{S}', \tag{2.72}$$

$$\mathcal{S} * \mathcal{E}' \subset \mathcal{S}, \qquad \mathcal{S} * \mathcal{S}' \subset \mathcal{E}, \tag{2.73}$$

and in all of these cases the convolution is separately continuous w.r.t. each of the two factors.

For instance, the inclusion $\mathcal{D}' * \mathcal{E}' \subset \mathcal{D}'$ is an extension of $L^1_{\text{loc}} * L^1_{\text{comp}} \subset L^1_{\text{loc}}$, and actually may be proved by approximating distributions by L^1_{loc}- or L^1_{comp}-functions, by using the latter property, and then passing to the limit. This procedure may also be used to prove $\mathcal{E}' * \mathcal{E}' \subset \mathcal{E}'$, too. The other inclusions may similarly be justified by approximation and passage to the limit.

3 The Fourier Transform in L^1

Integral Transforms. These are linear integral operators $\mathcal{T}$ that typically act on functions $\mathbb{R} \to \mathbb{C}$, and have the form

$$\big(\widehat{u}(\xi) := \big) \ (\mathcal{T}u)(\xi) = \int_{\mathbb{R}} K(\xi, x) u(x) \, dx \qquad \forall \xi \in \mathbb{R}, \tag{3.1}$$

for a prescribed kernel $K : \mathbb{R}^2 \to \mathbb{C}$, and for any transformable function u. [9] The main properties of this class of transforms include the following:

(i) Inverse Transform. Under appropriate restrictions, there exists another kernel $\widetilde{K} : \mathbb{R}^2 \to \mathbb{C}$ such that (formally)

$$\int_{\mathbb{R}} \widetilde{K}(x,\xi) K(\xi, y) \, d\xi = \delta_0(x-y) \qquad \forall x, y \in \mathbb{R}. \tag{3.2}$$

Denoting by R the integral operator associated to $\widetilde{K}$, we thus have $R\mathcal{T}u = \mathcal{T}Ru = u$ for any transformable u.

(ii) Commutation Formula. Any integral transform is associated to a class of linear operators (typically of differential type), that act on functions of time. For any such operator, L, there exists a function, $\widetilde{L}(\xi)$, such that

$$\mathcal{T} L \mathcal{T}^{-1} = \widetilde{L}(\xi) \quad \text{(this is a multiplicative operator)}. \tag{3.3}$$

[9] To devise hypotheses that encompass a large number of integral transforms is not easy and may not be convenient. In this brief overview we then refer to the Fourier transform. We are intentionally sloppy and drop regularity properties, that however are specified ahead.

By applying $\mathcal{T}$, an equation of the form $Lu = f$ (for a prescribed function $f = f(t)$) is then transformed into $\widetilde{L}(\xi)\widehat{u}(\xi) = \widehat{f}(\xi)$. Thus $\widehat{u} = \widehat{f}/\widetilde{L}$, whence $u = R(\widehat{f}/\widetilde{L})$. This procedure is at the basis of so-called *symbolic (or operational) calculus,* that was introduced by O. Heaviside at the end of the 19th century.

The first of the transforms that we illustrate is named after J. Fourier, who introduced it at the beginning of the 19th century, and is the keystone of all integral transforms. In the 1950s Laurent Schwartz introduced the space of *tempered distributions,* and extended the Fourier transform to this class. Because of the commutation formula, this transform allows one to reduce linear ordinary differential equations with constant coefficients to algebraic equations, and this found many uses in the study of stationary problems.

The Fourier Transform in L^1. We shall systematically deal with spaces of functions from the whole $\mathbb{R}^N$ to $\mathbb{C}$. We shall then write L^1 in place of $L^1(\mathbb{R}^N)$, C^0 in place of $C^0(\mathbb{R}^N)$, and so on. For any $u \in L^1$, we define the **Fourier transform** (also called **Fourier integral**) $\widehat{u}$ of u by [10]

$$\widehat{u}(\xi) := (2\pi)^{-N/2} \int_{\mathbb{R}^N} e^{-i\xi\cdot x} u(x)\, dx \qquad \forall \xi \in \mathbb{R}^N, \tag{3.4}$$

here $\xi \cdot x := \sum_{i=1}^N \xi_i x_i$.

Proposition 3.1 *The formula (3.4) defines a linear and continuous operator*

$$\begin{aligned} &\mathcal{F} : L^1 \to C^0_b : u \mapsto \widehat{u}; \\ &\|\widehat{u}\|_{L^\infty} \le (2\pi)^{-N/2} \|u\|_{L^1} \quad \forall u \in L^1. [Ex] \end{aligned} \tag{3.5}$$

(By C^0_b we denote the Banach space $C^0_b \cap L^\infty$.)

Thus $\widehat{u}_n \to \widehat{u}$ uniformly in $\mathbb{R}^N$ whenever $u_n \to u$ in L^1. In passing notice that $\|\widehat{u}\|_{L^\infty} = (2\pi)^{-N/2}\|u\|_{L^1}$ for any nonnegative $u \in L^1$, as in this case

$$\|\widehat{u}\|_{L^\infty} \le (2\pi)^{-N/2}\|u\|_{L^1} = \widehat{u}(0) \le \|\widehat{u}\|_{L^\infty}.$$

Apparently, no simple condition characterizes the image set $\mathcal{F}(L^1)$.

Proposition 3.2 *For any $u \in L^1$,* [11]

$$v(x) = u(x-y) \quad \Rightarrow \quad \widehat{v}(\xi) = e^{-i\xi\cdot y}\widehat{u}(\xi) \qquad \forall y \in \mathbb{R}^N, \tag{3.6}$$

$$v(x) = e^{ix\cdot\eta}u(x) \quad \Rightarrow \quad \widehat{v}(\xi) = \widehat{u}(\xi - \eta) \qquad \forall \eta \in \mathbb{R}^N, \tag{3.7}$$

$$v(x) = u\big(A^{-1}x\big) \quad \Rightarrow \quad \widehat{v}(\xi) = |\det A|\widehat{u}(A^*\xi) \quad \forall A \in \mathbb{R}^{N^2}, \det A \ne 0\,, \tag{3.8}$$

$$v(x) = \overline{u(x)} \quad \Rightarrow \quad \widehat{v}(\xi) = \overline{\widehat{u}(-\xi)}, \tag{3.9}$$

$$u \textit{ is even (odd, resp.)} \quad \Rightarrow \quad \widehat{u} \textit{ is even (odd, resp.)}, \tag{3.10}$$

$$u \textit{ is real and even} \quad \Rightarrow \quad \widehat{u} \textit{ is real (and even)}, \tag{3.11}$$

$$u \textit{ is real and odd} \quad \Rightarrow \quad \widehat{u} \textit{ is imaginary (and odd)}, \tag{3.12}$$

$$u \textit{ is radial} \quad \Rightarrow \quad \widehat{u} \textit{ is radial.} \tag{3.13}$$

[Ex]

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[10] Some authors introduce a factor 2π in the exponent under the integral, others omit the factor in front of the integral. Our definition is maybe the most frequently used. Each of these modifications simplifies some formulas, but none is able to simplify all of them.

[11] For any $A \in \mathbb{R}^{N^2}$, we set $(A^*)_{ij} := A_{ji}$ for any i, j. For any $z \in \mathbb{C}$, we denote its complex conjugate by $\overline{z}$. We say that u is **radial** iff $u(Ax) = u(x)$ for any x and any orthonormal matrix $A \in \mathbb{R}^{N^2}$ (i.e., with $A^* = A^{-1}$).

Henceforth by D (or D_j or D^α) we shall denote the operation of derivation in the sense of distributions.

Examples. (i) For any $A > 0$, if $u = \chi_{[-A,A]}$, then $\widehat{u}(\xi) = \sqrt{2/\pi}\,\frac{\sin(A\xi)}{\xi}$. [12]

(ii) We claim that

$$\begin{aligned} &u(x) = \exp(-a|x|^2) \qquad \forall x \in \mathbb{R}^N \quad \Rightarrow \\ &\widehat{u}(\xi) = (2a)^{-N/2} \exp(-|\xi|^2/(4a)) \qquad \forall \xi \in \mathbb{R}^N. \end{aligned} \tag{3.14}$$

Let us first prove this statement in the case of $a = 1/2$ and $N = 1$. [13] As $u' = -xu$ for any $x \in \mathbb{R}^N$,

$$i\xi\widehat{u}(\xi) \overset{(??)}{=} \widehat{D_x u}(\xi) = \widehat{-xu}(\xi) \overset{(??)}{=} (-i)D_\xi \widehat{u}(\xi),$$

that is, $D_\xi \widehat{u} = -\xi\widehat{u}$ for any $\xi \in \mathbb{R}^N$. As $u(0) = 1$ and

$$\widehat{u}(0) = (2\pi)^{-1/2} \int e^{-|x|^2/2}\, dx = 1,$$

$\widehat{u}$ solves the same Cauchy problem as u. Therefore for $N = 1$

$$u(x) = \exp(-|x|^2/2) \quad \Rightarrow \quad \widehat{u}(\xi) = \exp(-|\xi|^2/2) \quad \text{(i.e., } \widehat{u} = u). \tag{3.15}$$

For $N > 1$ and still for $a = 1/2$, we have $u(x) = \prod_{j=1}^N \exp(-x_j^2/2)$. Therefore

$$\begin{aligned} \widehat{u}(\xi) &= (2\pi)^{-N/2} \int e^{-\xi\cdot x} e^{-|x|^2/2}\, dx \\ &= \prod_{j=1}^N (2\pi)^{-1/2} \int e^{-\xi_j x_j} e^{-x_j^2/2}\, dx_j \overset{(3.15)}{=} \prod_{j=1}^N e^{-\xi_j^2/2} = e^{-|\xi|^2/2}. \end{aligned}$$

This concludes the proof of (3.14) for $a = 1/2$. The general formula then follows from (3.8).

Lemma 3.3 *Let $j \in \{1, ..., N\}$. If $\varphi, D_j\varphi \in L^1$ then $\int_{\mathbb{R}^N} D_j\varphi(x)\, dx = 0$.*

Proof. Let us define ρ as in (1.2), and set

$$\rho_n(x) := \rho(\frac{x}{n}) \quad \forall x \in \mathbb{R}^N, \forall n \in \mathbb{N}. \tag{3.16}$$

Hence $\rho_n(x) \to 1$ pointwise in $\mathbb{R}^N$ as $n \to \infty$, and

$$\Big|\int_{\mathbb{R}^N} \big[D_j\varphi(x)\big]\rho_n(x)\, dx\Big| = \Big|\int_{\mathbb{R}^N} \varphi(x) D_j\rho_n(x)\, dx\Big| \le \frac{1}{n}\|\varphi\|_{L^1} \cdot \|D_j\rho\|_\infty.$$

Therefore, by the dominated convergence theorem,

$$\int_{\mathbb{R}^N} D_j\varphi(x)\, dx = \lim_{n\to\infty} \int_{\mathbb{R}^N} \big[D_j\varphi(x)\big]\rho_n(x)\, dx = 0. \qquad \square$$

• **Proposition 3.4** *For any multi-index $\alpha \in \mathbb{N}^N$,*

$$u, D_x^\alpha u \in L^1 \quad \Rightarrow \quad (i\xi)^\alpha \widehat{u} = (D_x^\alpha u)\widehat{\ } \in C_b^0, \tag{3.17}$$

$$u, x^\alpha u \in L^1 \quad \Rightarrow \quad D_\xi^\alpha \widehat{u} = [(-ix)^\alpha u]\widehat{\ } \in C_b^0. \tag{3.18}$$

[12] Defining the *cardinal sinus* function $\operatorname{sinc} v := \frac{\sin v}{v}$ for any $v \in \mathbb{R}$, this also reads $\widehat{u}(\xi) = A\sqrt{2/\pi}\operatorname{sinc}(A\xi)$.

[13] A different proof of this result is based on integration along paths in the complex plane.

Proof. In both cases it suffices to prove the equality for any first-order derivative $D_j = \partial/\partial x_j$; the general case follows by induction.

(i) As

$$D_j[e^{-i\xi\cdot x}u(x)] = -i\xi_j e^{-i\xi\cdot x}u(x) + e^{-i\xi\cdot x}D_j u(x),$$

the integrability assumptions entail that $D_j[e^{-i\xi\cdot x}u(x)] \in L^1$. It then suffices to integrate the latter equality over $\mathbb{R}^N$, and to notice that $\int_{\mathbb{R}^N} D_j[e^{-i\xi\cdot x}u(x)]\,dx = 0$ by Lemma 3.3. Finally $(D_x^\alpha u)\widehat{\ } \in C_b^0$, by Proposition 3.1.

(ii) Denoting by e_j the unit vector in the jth direction, we have

$$\begin{aligned}\frac{\widehat{u}(\xi + te_j) - \widehat{u}(\xi)}{t} &= \int_{\mathbb{R}^N} \frac{e^{-i(\xi+te_j)\cdot x} - e^{-i\xi\cdot x}}{t} u(x)\,dx \\ &= \int_{\mathbb{R}^N} \frac{ix_j}{2} e^{-i(\xi+te_j/2)\cdot x}\sin(tx_j/2)u(x)\,dx.\end{aligned}$$

Passing to the limit as $t \to 0$, by the dominated convergence theorem we then get $D_j\widehat{u}(\xi) = -i(x_j u)\widehat{\ }(\xi)$ for any ξ. By Proposition 3.1, this is an element of C_b^0. □

Corollary 3.5 *Let* $m \in \mathbb{N}_0$.
(i) If $D_x^\alpha u \in L^1$ *for any* $\alpha \in \mathbb{N}_0^N$ *with* $|\alpha| \le m$, *then* $(1 + |\xi|)^m \widehat{u}(\xi) \in L^\infty$.
(ii) If $(1 + |x|)^m u \in L^1$, *then* $\widehat{u} \in C^m$. *[Ex]*

In other terms:
(i) the faster u decreases at infinity, the greater is the regularity of $\widehat{u}$;
(ii) the greater is the regularity of u, the faster $\widehat{u}$ decreases at infinity.

Proposition 3.6 (Riemann-Lebesgue) *For any* $u \in L^1$, $\widehat{u}(\xi) \to 0$ *as* $|\xi| \to +\infty$, *and* $\widehat{u}$ *is uniformly continuous in* $\mathbb{R}^N$.

Proof. For any $u \in L^1$, there exists a sequence $\{u_n\}$ in $\mathcal{D}$ such that $u_n \to u$ in L^1. By part (i) of Corollary 3.5, $\widehat{u}_n(\xi) \to 0$ as $|\xi| \to +\infty$. This holds also for $\widehat{u}$, as $\widehat{u}_n \to \widehat{u}$ uniformly in $\mathbb{R}^N$ by Proposition 3.1. [14] As $\widehat{u} \in C_b^0$, the uniform continuity follows. □

Theorem 3.7 (Parseval) *The formal adjoint of* $\mathcal{F}$ *coincides with* $\mathcal{F}$ *itself, that is,*

$$\int_{\mathbb{R}^N} \widehat{u}\, v\,dx = \int_{\mathbb{R}^N} u\,\widehat{v}\,dx \qquad \forall u, v \in L^1. \tag{3.19}$$

Moreover,

$$u * v \in L^1, \quad \text{and} \quad (u*v)\widehat{\ } = (2\pi)^{N/2}\widehat{u}\,\widehat{v} \qquad \forall u, v \in L^1. \tag{3.20}$$

Proof. By the theorems of Tonelli and Fubini, for any $u, v \in L^1$ we have

$$\int_{\mathbb{R}^N} \widehat{u}(y)v(y)\,dy = (2\pi)^{-N/2} \iint_{\mathbb{R}^N\times\mathbb{R}^N} e^{-iy\cdot x}u(x)v(y)\,dx\,dy = \int_{\mathbb{R}^N} u(y)\widehat{v}(y)\,dy.$$

[14] Here is an alternative argument. By direct evaluation of the integral one may check that the assertion holds for the characteristic function of any N-dimensional interval $[a_1, b_1] \times \cdots \times [a_N, b_N]$. It then suffices to approximate u in L^1 by a sequence of finite linear combinations of characteristic functions of N-dimensional intervals.

On the other, by the change of integration variable $z = x - y$,

$$(u*v)\widehat{}(\xi) = (2\pi)^{-N/2} \iint_{\mathbb{R}^N \times \mathbb{R}^N} e^{-2\pi i \xi \cdot x} u(x-y) v(y)\, dx\, dy$$
$$= (2\pi)^{-N/2} \int_{\mathbb{R}^N} e^{-2\pi i \xi \cdot z} u(z)\, dz \int_{\mathbb{R}^N} e^{-2\pi i \xi \cdot y} v(y)\, dy$$
$$= (2\pi)^{N/2} \widehat{u}(\xi) \widehat{v}(\xi). \qquad \square$$

Next we present the inversion formula for the Fourier transform. First let us introduce the so-called *conjugate Fourier transform:*

$$\widetilde{\mathcal{F}}(v)(x) := (2\pi)^{-N/2} \int_{\mathbb{R}^N} e^{2\pi i \xi \cdot x} v(\xi) d\xi \qquad \forall v \in L^1, \forall x \in \mathbb{R}^N. \tag{3.21}$$

This operator differs from $\mathcal{F}$ just in the sign of the imaginary unit. Obviously, $\widetilde{\mathcal{F}} v = \overline{\mathcal{F} \bar{v}}$ for any $v \in L^1$. Clearly the properties of $\widetilde{\mathcal{F}}$ are analogous to those of $\mathcal{F}$.

Theorem 3.8 *For any $u \in L^1 \cap C^0 \cap L^\infty$, if $\widehat{u} \in L^1$ then*

$$u(x) = \widetilde{\mathcal{F}}(\widehat{u})(x) \qquad \forall x \in \mathbb{R}^N. \tag{3.22}$$

Proof. Let us set $v(x) := \exp(-|x|^2/2)$ for any $x \in \mathbb{R}^N$. By the Tonelli and Fubini theorems, we have

$$\int_{\mathbb{R}^N} \widehat{u}(\xi) v(\xi) e^{i\xi \cdot x}\, d\xi = (2\pi)^{-N/2} \iint_{\mathbb{R}^N \times \mathbb{R}^N} u(y) e^{-i\xi \cdot y} v(\xi) e^{i\xi \cdot x}\, dy\, d\xi$$
$$= \int_{\mathbb{R}^N} u(y) \widehat{v}(y - x)\, dy = \int_{\mathbb{R}^N} u(x+z) \widehat{v}(z)\, dz \qquad \forall x \in \mathbb{R}^N.$$

Let us now replace $v(\xi)$ by $v_\varepsilon(\xi) := v(\varepsilon \xi)$, for any $\varepsilon > 0$. By (3.8), $\widehat{v_\varepsilon}(z) = \varepsilon^{-N} \widehat{v}(\varepsilon^{-1} z)$; by a further change of variable of integration we then get

$$\int_{\mathbb{R}^N} \widehat{u}(\xi) v(\varepsilon \xi) e^{i\xi \cdot x}\, d\xi = \int_{\mathbb{R}^N} u(x + \varepsilon y) \widehat{v}(y)\, dy \qquad \forall x \in \mathbb{R}^N.$$

As u is continuous and bounded, by the dominated convergence theorem we may pass to the limit under integral as $\varepsilon \to 0$. We thus get

$$v(0) \int_{\mathbb{R}^N} \widehat{u}(\xi) e^{i\xi \cdot x}\, d\xi = u(x) \int_{\mathbb{R}^N} \widehat{v}(y)\, dy. \tag{3.23}$$

On the other hand, by (3.14) and by the classical Poisson formula,

$$\int_{\mathbb{R}^N} \widehat{v}(y)\, dy = \int_{\mathbb{R}^N} \exp(-|y|^2/2)\, dy = \left(\int_{\mathbb{R}} \exp(-|y|^2/2)\, dy \right)^N = (2\pi)^{N/2}.$$

As $v(0) = 1$, (3.22) then follows from (3.23). $\square$

By Proposition 3.1, for the above argument the regularity assumptions of Theorem 3.8 are actually needed, as $\bar{u} = \mathcal{F}(\overline{\widehat{u}})$. However, by a more refined argument one might show that (3.22) holds under the only hypotheses that $u, \widehat{u} \in L^1$. (Of course, a posteriori one then gets that $u, \widehat{u} \in C^0_b$.)

By Theorem 3.8, $\mathcal{F}(u) \equiv 0$ only if $u \equiv 0$; hence the Fourier transform $L^1 \to C^0_b$ is injective. Under the assumptions of this theorem, we also have

$$\overline{\widehat{\widehat{u}}(x)} = \bar{u}(-x) \qquad \forall x \in \mathbb{R}^N. \tag{3.24}$$

We recall that $B(0, R)$ denotes the closed ball in $\mathbb{R}^N$ with center at the origin and radius R.

Theorem 3.9 (Paley-Wiener) *For any $u \in C^\infty(\mathbb{R}^N)$ and any $R > 0$, $\operatorname{supp} u \subset B(0,R)$ iff $\mathcal{F}(u)$ can be extended to a holomorphic function $\mathbb{C}^N \to \mathbb{C}$ (also denoted by $\mathcal{F}(u)$)* [15] *such that*

$$\forall m \in \mathbb{N}, \exists C \geq 0 : \forall z \in \mathbb{C}^N \qquad |[\mathcal{F}(u)](z)| \leq C \frac{e^{R|\mathcal{I}(z)|}}{(1+|z|)^m}. [] \tag{3.25}$$

This extended function $\mathcal{F}(u) : \mathbb{C}^N \to \mathbb{C}$ is called the **Fourier-Laplace transform** of u, for reasons that will appear clear ahead.

Overview of the Fourier Transform in L^1. We defined the classic Fourier transform $\mathcal{F} : L^1 \to C^0_b$ and derived its basic properties. In particular, we saw that

(i) $\mathcal{F}$ transforms partial derivatives to multiplication by powers of the independent variable (up to a multiplicative constant) and conversely. This is at the basis of the application of the Fourier transform to the study of linear partial differential equations with constant coefficients, that we shall outline ahead.

(ii) $\mathcal{F}$ establishes a correspondence between the regularity of u and the order of decay of $\widehat{u}$ at ∞, and conversely between the order of decay of u at ∞ and the regularity of $\widehat{u}$. In the limit case of a compactly supported function, its Fourier transform may be extended to an entire holomorphic function $\mathbb{C}^N \to \mathbb{C}$.

(iii) $\mathcal{F}$ transforms the convolution of two functions to the product of their transforms (the converse statement may fail, because of summability restrictions).

(iv) Under suitable regularity restrictions, the inverse transform exists, and has an integral representation analogous to that of the direct transform.

The properties of the two transforms are then similar; this accounts for the duality of the statements (i) and (ii). However the assumptions are not perfectly symmetric; in the next section we shall see a different functional framework where this is remedied.

The inversion formula (3.22) also provides an interpretation of the Fourier transform. (3.22) represents u as a weighted average of the *harmonic components* $x \mapsto e^{i\xi\cdot x}$. For any $\xi \in \mathbb{R}^N$, $\widehat{u}(\xi)$ is the *amplitude* of the component having *vector frequency* ξ (that is, frequency ξ_i in each direction x_i). Therefore any function which fulfills (3.22) may equivalently be represented by specifying either the value $u(x)$ at a.a. points $x \in \mathbb{R}^N$, or the amplitude $\widehat{u}(\xi)$ for a.a. frequencies $\xi \in \mathbb{R}^N$. (Loosely speaking, any non-identically vanishing $u \in \mathcal{D}$ has *harmonic components* of arbitrarly large frequencies.)

The analogy between the Fourier transform and the Fourier series is obvious, and will be briefly discussed at the end of the next section.

4 Extensions of the Fourier Transform

Fourier Transform of Measures. The Fourier transform may easily be extended to any finite complex Borel measure μ on $\mathbb{R}^N$, simply by replacing $f(x)\,dx$ with $d\mu(x)$ in (3.4). In this case one speaks of the **Fourier-Stieltjes transform.** Most of the previously established properties holds also in this case. For instance, transformed functions are still elements of C^0_b and fulfill the properties of transformation of derivatives and multiplication by a power of x. The Riemann-Lebesgue theorem (Proposition 3.6) however fails; e.g., $\widehat{\delta_0}(\xi)$ does not vanish as $|\xi| \to +\infty$. The transform of the Dirac measure at the origin actually coincides with the function identically equal to $(2\pi)^{-N/2}$, that is, $\widehat{\delta_0} = (2\pi)^{-N/2}$.

[15] A function $\mathbb{C}^N \to \mathbb{C}$ is called holomorphic (or analytic) iff it is separately olomorphic with respect to each variable. For any $z \in \mathbb{C}^N$, we set $|z| = \left(\sum_{i=1}^N |z_i|^2\right)^{1/2}$ and $\mathcal{I}(z) = (\mathcal{I}(z_1), ..., \mathcal{I}(z_N))$ (the vector of the imaginary parts).

Fourier Transform in $\mathcal{S}$. For any $u \in \mathcal{D}$, by Theorem 3.9 $\widehat{u}$ is holomorphic; hence $\widehat{u} \in \mathcal{D}$ only if $\widehat{u} \equiv 0$, namely $u \equiv 0$. Thus $\mathcal{D}$ is not stable by Fourier transform. This means that the set of the frequencies of the harmonic components of any non-identically vanishing $u \in \mathcal{D}$ is unbounded. This situation induced L. Schwartz to introduce the space of rapidly decreasing functions $\mathcal{S}$, cf. Sect. 1, and to extend the Fourier transform to this space and to its (topological) dual. Next we review the tenets of that theory.

Proposition 4.1 *(The restriction of) $\mathcal{F}$ operates in $\mathcal{S}$ and is continuous. Moreover, the formulae of Proposition 3.4 and Theorem 3.7 hold in $\mathcal{S}$ without any restriction, $\mathcal{F}$ is invertible in $\mathcal{S}$, and $\mathcal{F}^{-1} = \widetilde{\mathcal{F}}$ (cf. (3.22)). [Ex]*

The first part is easily checked by repeated use of the Leibniz rule, because of the stability of the space $\mathcal{S}$ w.r.t. multiplication by any polynomial and w.r.t. application of any differential operator (with constant coefficients). Actually, $\mathcal{S}$ is the smallest space that contains L^1 and has these properties. [Ex] The next statement extends and also completes (3.20).

Proposition 4.2 *For any $u, v \in \mathcal{S}$,*

$$u{*}v \in \mathcal{S}, \qquad (u{*}v)\widehat{\ } = (2\pi)^{N/2}\widehat{u}\,\widehat{v} \quad \text{in } \mathcal{S}, \tag{4.1}$$

$$uv \in \mathcal{S}\,, \qquad (uv)\widehat{\ } = (2\pi)^{-N/2}\widehat{u}{*}\widehat{v} \quad \text{in } \mathcal{S}. \tag{4.2}$$

Proof. The first statement is a direct extension of (3.20). Let us prove the second one.

It is easily checked that $uv \in \mathcal{S}$. By writing (3.20) with $\widehat{u}$ and $\widehat{v}$ in place of u and v, and $\widetilde{\mathcal{F}}$ in place of $\mathcal{F}$, we have

$$\widetilde{\mathcal{F}}(\widehat{u}{*}\widehat{v}) = (2\pi)^{N/2}\widetilde{\mathcal{F}}(\widehat{u})\,\widetilde{\mathcal{F}}(\widehat{v}) = (2\pi)^{N/2}u\,v.$$

By applying $\mathcal{F}$ to both members of this equality, (4.2) follows. □

Fourier Transform in $\mathcal{S}'$. Denoting by $\mathcal{F}^\tau$ the transposed of $\mathcal{F}$, we set [16]

$$\bar{\mathcal{F}} := [\mathcal{F}^\tau]^* : \mathcal{S}' \to \mathcal{S}'. \tag{4.3}$$

By the Parseval Theorem 3.7, $\mathcal{F}^\tau = \mathcal{F}$; hence $\bar{\mathcal{F}} = \mathcal{F}^*$, that is,

$$\langle \bar{\mathcal{F}}(T), v\rangle := \langle T, \mathcal{F}(v)\rangle \qquad \forall v \in \mathcal{S}, \forall T \in \mathcal{S}'. \tag{4.4}$$

As $\mathcal{S}$ is sequentially dense in $\mathcal{S}'$, $\bar{\mathcal{F}}$ is the unique continuous extension of the Fourier transform from $\mathcal{S}$ to $\mathcal{S}'$.

Henceforth we shall use the same symbols $\mathcal{F}$ or $\widehat{\ }$ for the many restrictions and extensions of the Fourier transform. We shall thus write $\mathcal{F}(T)$, or $\widehat{T}$, in place of $\bar{\mathcal{F}}(T)$.

Proposition 4.3 *$\mathcal{F}$ may be uniquely extended to an operator which acts in $\mathcal{S}'$ and is continuous. Moreover, the formulae of Proposition 3.4 and Theorem 3.7 hold in $\mathcal{S}'$ without any restriction, $\mathcal{F}$ is invertible in $\mathcal{S}'$, and $\mathcal{F}^{-1} = \widetilde{\mathcal{F}}$. [Ex]*

For instance, for any $v \in \mathcal{D}$ we have

$$\begin{aligned} {}_{\mathcal{D}'}\langle i^{|\alpha|}\xi^\alpha \widehat{T}, v\rangle_{\mathcal{D}} &= {}_{\mathcal{D}'}\langle \widehat{T}, i^{|\alpha|}\xi^\alpha v\rangle_{\mathcal{D}} = {}_{\mathcal{D}'}\langle T, [i^{|\alpha|}\xi^\alpha v]\widehat{\ }\rangle_{\mathcal{D}} \\ &\overset{(3.18)}{=} {}_{\mathcal{D}'}\langle T, (-D)^\alpha \widehat{v}\rangle_{\mathcal{D}} = {}_{\mathcal{D}'}\langle D^\alpha T, \widehat{v}\rangle_{\mathcal{D}} = {}_{\mathcal{D}'}\langle (D^\alpha T)\widehat{\ }, v\rangle_{\mathcal{D}}. \end{aligned}$$

As $\mathcal{D}$ is a dense subspace of $\mathcal{S}$, we conclude that

$$i^{|\alpha|}\xi^\alpha \widehat{T} = (D_x^\alpha T)\widehat{\ } \in \mathcal{S}' \qquad \forall T \in \mathcal{S}', \forall \alpha \in \mathbb{N}^N. \tag{4.5}$$

[16] Notice that $\mathcal{F}: \mathcal{S} \to \mathcal{S}$, $\mathcal{F}^\tau : \mathcal{S} \to \mathcal{S}$, $\mathcal{F}^* : \mathcal{S}' \to \mathcal{S}'$.

Proposition 4.4 *For any $u \in \mathcal{S}$ and any $T \in \mathcal{S}'$,*

$$u{*}T \in \mathcal{S}', \qquad (u{*}T)\widehat{\ } = (2\pi)^{N/2}\widehat{u}\,\widehat{T} \quad \text{in } \mathcal{S}', \tag{4.6}$$

$$uT \in \mathcal{S}', \qquad (uT)\widehat{\ } = (2\pi)^{-N/2}\widehat{u}{*}\widehat{T} \quad \text{in } \mathcal{S}'. \tag{4.7}$$

Next we extend to $\mathcal{E}'$ the Fourier-Laplace transform of the previous section.

Theorem 4.5 *Let us set*

$$\widehat{T}(\xi) = {}_{\mathcal{E}'}\langle T, e^{-ix\cdot\xi}\rangle_{\mathcal{E}} \qquad \forall T \in \mathcal{E}', \forall \xi \in \mathbb{R}^N. \tag{4.8}$$

This expression may be extended to any $\xi \in \mathbb{C}^N$, and is a holomorphic function.

Proof. For any $n \in \mathbb{N}$, let us define the mollifier ρ_n as in (3.16), and set $(T*\rho_n)(x) := \langle T_y, \rho_n(x-y)\rangle$ for any $x \in \mathbb{R}^N$. (The index y indicates that T acts on the variable y; on the other hand here x is just a parameter.) This yields $T * \rho_n \to T$ in $\mathcal{E}'$, hence also in $\mathcal{S}'$ as $\mathcal{E}' \subset \mathcal{S}'$ with continuous and sequentially dense injections. Therefore

$$(T * \rho_\varepsilon)\widehat{\ } \to \widehat{T} \qquad \text{in } \mathcal{S}'. \tag{4.9}$$

On the other hand, as $T * \rho_n \in \mathcal{E}$ and $\int_{\mathbb{R}^N} \rho_n(x)\,dx = 1$, we have

$$\begin{aligned}(T * \rho_n)\widehat{\ }(\xi) &= (2\pi)^{-N/2} \int_{\mathbb{R}^N} e^{-i\xi\cdot x}(T * \rho_n)(x)\,dx \\ &= (2\pi)^{-N/2} \int_{\mathbb{R}^N\times\mathbb{R}^N} e^{-i\xi\cdot x}\langle T_y, \rho_n(x-y)\rangle\,dx\,dy \\ &= (2\pi)^{-N/2}\langle T_y, e^{-i\xi\cdot y}\int_{\mathbb{R}^N} e^{-i\xi\cdot(x-y)}\rho_n(x-y)\,dx\rangle = \langle T_y, e^{-i\xi\cdot y}\rangle\widehat{\rho}_n(\xi),\end{aligned}$$

and this is a holomorphic function of ξ. As $\varepsilon \to 0$, $\widehat{\rho}_n(\xi) \to 1$ uniformly on any compact subset of $\mathbb{R}^N$. Therefore

$$(T * \rho_n)\widehat{\ }(\xi) = \langle T_y, e^{-i\xi\cdot y}\rangle\widehat{\rho}_n(\xi) \to \langle T_y, e^{-i\xi\cdot y}\rangle \qquad \text{in } \mathcal{S}'.$$

By (4.9) we then conclude that $\widehat{T}(\xi) = \langle T_y, e^{-2\pi i\xi\cdot y}\rangle$ for any $\xi \in \mathbb{R}^N$, and this function is holomorphic. □

Theorem 4.6 (Paley-Wiener-Schwartz) *For any $T \in \mathcal{S}'$ and any $R > 0$,* supp $T \subset B(0,R)$ *iff $\mathcal{F}(T)$ may be extended to a holomorphic function $\mathbb{C}^N \to \mathbb{C}$ (also denoted by $\mathcal{F}(T)$) such that*

$$\exists m \in \mathbb{N}_0, \exists C \geq 0 : \forall z \in \mathbb{C}^N \qquad |[\mathcal{F}(T)](z)| \leq C(1+|z|)^m e^{R|\mathcal{I}(z)|}. \qquad [] \tag{4.10}$$

(Thus $\mathcal{F}$ maps $\mathcal{E}'$ to holomorphic functions.)

We can now further extend (4.6) as follows.

Proposition 4.7 *For any $u \in \mathcal{E}'$ and any $T \in \mathcal{S}'$,*

$$u{*}T \in \mathcal{S}', \qquad (u{*}T)\widehat{\ } = (2\pi)^{N/2}\widehat{u}\,\widehat{T} \quad \text{in } \mathcal{S}', \tag{4.11}$$

Notice that $\widehat{u}\widehat{T} \in \mathcal{S}'$, as by the latter theorem $\widehat{u}$ is holomorphic.

Fourier Transform in L^2. As $L^2 \subset \mathcal{S}'$, any function of L^2 has a Fourier transform. Next we study the restriction of $\mathcal{F}$ to L^2 and show that it is an isometry.

• **Theorem 4.8 (Plancherel)**

$$u \in L^2 \quad \Leftrightarrow \quad \widehat{u} \in L^2 \qquad \forall u \in \mathcal{S}'. \tag{4.12}$$

The restriction of $\mathcal{F}$ to L^2 is an isometry:

$$\|\widehat{u}\|_{L^2} = \|u\|_{L^2} \qquad \forall u \in L^2. \tag{4.13}$$

Moreover, for any $u \in L^2$,

$$(2\pi)^{-N/2} \int_{]-R,R[^N} e^{-i\xi\cdot x} u(x)\,dx \;\to\; \widehat{u}(\xi) \qquad \textit{in } L^2, \textit{ as } R \to +\infty. \tag{4.14}$$

Therefore this sequence also converges in measure on any bounded subset of $\mathbb{R}^N$; it also converges a.e. in $\mathbb{R}^N$, as $R \to +\infty$ along some sequence which may depend on u.

Proof. For any $u \in \mathcal{S}$, we know that $\widehat{u} \in \mathcal{S}$. Moreover, by (3.19) and (3.24),

$$\int_{\mathbb{R}^N} |\widehat{u}|^2\,dx = \int_{\mathbb{R}^N} \widehat{u}\bar{\widehat{u}}\,dx = \int_{\mathbb{R}^N} u\hat{\bar{\widehat{u}}}\,dx = \int_{\mathbb{R}^N} u(x)\overline{\hat{\widehat{u}}(-x)}\,dx = \int_{\mathbb{R}^N} u\bar{u}\,dx$$
$$= \int_{\mathbb{R}^N} |u|^2\,dx.$$

Therefore, as $\mathcal{S} \subset L^2$ with continuity and density, the restriction of $\mathcal{F}$ to L^2 is an isometry with respect to the L^2-metric. Hence $\mathcal{F}$ maps L^2 to itself.

In order to prove (4.14), for any $R > 0$ and any $x \in \mathbb{R}$, let us set $\chi_R(x) := 1$ if $|x_i| \le R$ for $i = 1, ..., N$, and $\chi_R(x) := 0$ otherwise. Then $u\chi_R \in L^1 \cap L^2$ and $u\chi_R \to u$ in L^2. Hence, by (4.13),

$$(2\pi)^{-N/2} \int_{]-R,R[^N} e^{-i\xi\cdot x} u(x)\,dx = (2\pi)^{-N/2} \int_{\mathbb{R}^N} e^{-i\xi\cdot x} u(x)\chi_R(x)\,dx$$
$$= (u\chi_R)\widehat{\ }(\xi) \;\to\; \widehat{u}(\xi) \qquad \text{in } L^2. \qquad \square$$

Remarks. (i) We saw that in any Hilbert space H the scalar product is determined by the norm: $2(u,v) = \|u+v\|^2 - \|u\|^2 - \|v\|^2$ for any $u, v \in H$. (4.13) then entails that

$$\int_{\mathbb{R}^N} u(x)\bar{v}(x)\,dx = \int_{\mathbb{R}^N} \widehat{u}(\xi)\overline{\widehat{v}(\xi)}\,d\xi \qquad \forall u, v \in L^2, \tag{4.15}$$

(ii) The representation (4.14) is more general:

$$\int_{]-R,R[^N} e^{-2\pi i\xi\cdot x} u(x)\,dx \;\to\; \widehat{u}(\xi) \qquad \text{in } \mathcal{S}', \text{ as } R \to +\infty\,, \forall u \in \mathcal{S}' \cap L^1_{\rm loc}\,. \tag{4.16}$$

The above argument also allow one to generalize the inversion Theorem 3.8.

(iii) The Lebesgue-integral representation (3.4) is meaningful only if $u \in L^1$. Anyway it may be useful to know of cases in which the (extended) Fourier transform maps functions to functions. We claim that, for any $p \in [1,2]$, any function $u \in L^p$ may be written as the sum of a function of L^1 and one of L^2, i.e.,

$$L^p \subset L^1 + L^2 \qquad \forall p \in [1,2]\,. \tag{4.17}$$

Indeed, setting $\chi := 1$ where $|u| \ge 1$ and $\chi := 0$ elsewhere, we have $u\chi \in L^1$, $u(1-\chi) \in L^2$ and $u = u\chi + u(1-\chi)$. [17] Therefore, as $\mathcal{F} : L^1 \to L^\infty$ and $\mathcal{F} : L^2 \to L^2$,

$$\mathcal{F}(u) = \mathcal{F}(u\chi) + \mathcal{F}(u(1-\chi)) \in L^\infty + L^2 \qquad \forall u \in L^p, \forall p \in [1,2]\,. \tag{4.18}$$

[17] Similarly one can show that $L^p \subset L^q + L^r$ whenever $1 \le q < p < r \le \infty$.

Thus $\mathcal{F}(u)$ is a regular distribution, although it may admit no integral representation.

This is made more precise by the following result, which is a direct consequence of the classic Riesz-Thorin Theorem 2.4.

Theorem 4.9 (Hausdorff-Young) [18] *Let $p \in [1,2]$, $p' := p/(p-1)$ if $p > 1$, and $p' = \infty$ if $p = 1$. Then (the restriction of) $\mathcal{F}$ is a linear and continuous operator $L^p \to L^{p'}$. More precisely, for any $u \in L^p$ we have $\widehat{u} \in L^{p'}$ and $\|\widehat{u}\|_{L^{p'}} \le \|u\|_{L^p}$.*

Proof. As $\mathcal{F}$ is linear and continuous as an operator $L^1 \to L^\infty$ and $L^2 \to L^2$, it suffices to apply the classical Riesz-Thorin Theorem 2.4. □

Because of the symmetry between the definitions of the direct and inverse Fourier transform, see formulas (3.4) and (3.22), the results that we established for $\mathcal{F}$ in $\mathcal{S}$ and in $\mathcal{S}'$, in particular Theorems 4.1 and 4.8, hold also for $\mathcal{F}^{-1}$.

We claim that $\mathcal{F}$ does not map L^p to L^q for any $q \neq p'$. Let $u \in L^p$ be such that $\mathcal{F}(u) \in L^q$. For any $\lambda > 0$, setting $u_\lambda(x) := u(\lambda x)$ for any x, by (3.8) we have $\mathcal{F}(u_\lambda) = \lambda^{-N}\mathcal{F}(u)_{1/\lambda}$; hence

$$\frac{\|\mathcal{F}(u_\lambda)\|_{L^q}}{\|u_\lambda\|_{L^p}} = \lambda^{-N}\frac{\|\mathcal{F}(u)_{1/\lambda}\|_{L^q}}{\|u_\lambda\|_{L^p}} = \lambda^{N(-1+1/q+1/p)}\frac{\|\mathcal{F}(u)\|_{L^q}}{\|u\|_{L^p}} \tag{4.19}$$

and this ratio is uniformly bounded w.r.t. λ iff $q = p'$.

Overview of the Extensions of the Fourier Transform. At first we noticed that the Fourier transform (3.5) has an obvious extension for any complex Borel measure μ; loosely speaking, this is just defined by replacing $u(x)dx$ by $d\mu$ in the definition (3.4). By the Paley-Wiener theorem, $\mathcal{D}$ is not stable under application of the Fourier transform. However, $\mathcal{F}$ maps the Schwartz space $\mathcal{S}$ to itself; this allowed us to extend $\mathcal{F}$ to $\mathcal{S}'$ by transposition. We also saw that $\mathcal{F}$ is an isometry in L^2 (Plancherel theorem), that in this space $\mathcal{F}$ admits an integral representation as a principal value, and that $\mathcal{F}$ is also linear and continuous from L^p to $L^{p/(p-1)}$, for any $p \in \,]1,2[$.

Finally, we saw that the Fourier series arise as Fourier transforms of periodic functions.

Note: The Fourier transform is a homomorphism from the algebra $(L^1, *)$ to the algebra $(L^\infty, \cdot)$ (here "$\cdot$" stands for the product a.e.), and is an isomorphism between the algebras $(\mathcal{S}, *)$ and $(\mathcal{S}, \cdot)$; cf. (3.20).

[18] We set $\infty^{-1} := 0$.