Filters

NOTA: Questa parte e' ancora allo stato di bozza, ma e' sufficiente ai fini della preparazione dell'esame.

Symbology. • indicates important results. $*$ indicates complements.
[Ex] means that the proof is left as exercise. [] means that the proof is missing.

1 Linear Systems

In this chapter we outline some elements of the theory of linear systems, in an effort to combine a mathematical approach with the point of view and the language of *signal processing* and *control theory.* These disciplines have been extensively developed by engineers, and in addressing them the mathematician is called to formulate several notions and results in a rigorous form.

First, we shall introduce some basic concepts like time-invariance, causality, stability, and so on. We shall relate these properties to the response of the system to the unit impulse, and shall reformulate the action of time-invariant systems as the convolution between the input function and the impulse response. Via Fourier or Laplace transform, we shall also transform the input-output relation from the space of time to that of frequency. [1] We shall see that systems can be arranged in series, in parallel, and in retroaction (or feedback).

1.1 Signals and systems

The notion of *system* occurs in many scientific and technical disciplines: physics, chemistry, biology, economics, control theory, engineering of telecommunications, electrical engineering, and so on. For our aims, we define a system as a unitary whole, that can exchange information with the exterior. We shall be concerned with *transmission systems,* in which an input of information produces an output of information. Such a system will be regarded as a *black box:* by this we mean that attention will be paid just to the input and output, without considering the internal structure of the system (which will appear *black* to our eyes).

We can define a *signal* as a function of time that represents the evolution of a physical quantity. Examples include: voltage and current of an electric circuit in electromagnetism, tension and deformation of a string in continuum mechanics, air pressure and sound in acoustics, and so on. In this context the system is represented by an operator that transforms an input signal to an output signal. For instance, in a control system the input is a *control* (that is, a quantity that we can control), and the output is an *observable* (that is, a quantity that we can observe). Here we shall just deal with complex-valued signals and linear systems.

Let us consider a system that maps a time-dependent complex scalar input signal to a time-dependent complex scalar output. [2] One distinguishes between discrete and *analogical* systems, in which time is discrete or continuous, respectively, and is represented by $n \in \mathbb{Z}$ or $t \in \mathbb{R}$. Moreover, there are *digital* systems, in which input and output are discrete *(quantized)* variables. One also

[1] As we did above, when we redefined the Fourier transform in Section XXX, by frequency we refer to the number of cycles in the unit time. On the other hand, when dealing with the Laplace transform, by frequency we shall refer to the complex frequency.

[2] In the engineering literature this is called a SISO *(single input and single output)* system. We shall not deal with MIMO *(multiple input and multiple output)* systems. In this second case the scalar transfer function (see ahead) is replaced by a matrix-valued transfer function, and the analysis is similar to the present one. (BIBO is a different animal that we shall encounter ahead...)

distinguishes between *deterministic* and *stochastic* systems. In this presentation we shall just deal with deterministic analogical systems.

We shall constrain input and output signals to appropriate topological linear spaces [3] of complex-valued functions of time, with a space Φ of *admissible inputs* and a space Ψ of outputs. [4] We shall then associate each system with a linear operator $L : \Phi \to \Psi : f \mapsto g$, which we shall often identify with the system itself. [5] The space Φ may include $L^p(\mathbb{R})$ for some $p \in [1, \infty]$, $C^0_b(\mathbb{R})$, and also Dirac masses. In several cases we can assume $\Psi = \mathcal{D}'(\mathbb{R})$,

Examples. Here are some linear systems that are studied in physics and engineering.

(i) *Amplifiers,* that are represented by the *state equation* $L(f) = Af$, with *amplification factor* $A > 0$. These systems are called *amplifiers in strict sense* if $A > 1$, *attenuators* if $A < 1$.

(ii) *Delayed systems,* with state equation

$$L(f) = f(\cdot - a) =: f_a \qquad \text{for some fixed } a > 0. \tag{1.1}$$

(iii) *Derivators,* that are characterized by $L(f) = f'$ (the time derivative).

(iv) *Integrators,* that are characterized by $[L(f)](t) = \int_{-\infty}^{t} f(s)\, ds$.

(v) *RLC circuits.* These are models of electrical systems, and are constructed by combining either in series or in parallel (see ahead) elementary linear constituents: resistors, inductors, condensers [6] and an electrical generator, which typically generates an alternating-current. If these elements are connected in series, the system is characterized by an ODE of the form

$$Ly''(t) + Ry'(t) + \frac{y(t)}{C} = f(t). \tag{1.2}$$

Here by $f = f(t)$ we denote an applied tension (the input), by $y = y(t)$ the electric current flowing in the system (the output), by L the inductance, by R the resistance, by C the capacitance. This is known as the equation of the *damped driven harmonic oscillator* (the basic equation of the harmonic oscillator reads $ay'' + y = 0$, with $a > 0$). This model is ubiquitous in physics, and has a huge number of applications.

(vi) *Rheological models.* These are *lumped models* [7] that are used to describe the mechanical behaviour of continuous materials. They represent lumped mechanical systems that consist in a mass, a spring and a dashpot (or damper, i.e., a viscous element). The corresponding state equation is an ODE of the form (1.2):

$$my''(t) + \nu y'(t) + ky(t) = f(t). \tag{1.3}$$

Here by $f = f(t)$ we denote an applied force (the input), by $y = y(t)$ the displacement (the output), by m the mass, by ν the viscosity, by k the compliance of the spring.

[3] This means that they are linear spaces as well as topological spaces, and that the linear operations (namely sum and multiplication by a scalar) are continuous with respect to that topology. This includes all function spaces used in mathematics.

[4] As above, we still just write $\mathcal{D}'$, omitting to display $\mathbb{R}$.

[5] Mind the ℓs: L is the system, L^p is the Lebesgue space, $\mathcal{L}$ is the Laplace transform...

[6] In the technical literature, these are often referred to as *passive electric elements,* since they generate no energy. Resistors dissipate (electric) energy, whereas inductors. and condensers first store and then deliver it. Of course, here we refer to ideal elements, since real devices combine these different behaviours.

[7] They correspond to ODEs, whereas (space-) *distributed models* correspond to PDEs.

Acoustic models are also analogous to rheological models. A basic example is the *Helmholtz resonator,* which is represented by a state equation analogous to (1.3). [8]

(vii) More general linear *differential systems.* Denoting the input by f and the output by $y = L(f)$ as above, these systems are characterized by a state equation of the form

$$P(t, D)y = Q(t, D)f, \tag{1.4}$$

with $P(t, D)$ and $Q(t, D)$ time-dependent polynomial differential operators. This equation accounts for a large class of systems used in applications. In particular P and Q may have constant coefficients. [9] The degree of P is called the *order of the system:* for instance harmonic oscillators have order two.

Here we shall not consider nonlinear systems.

1.2 Main properties

Next we outline some relevant properties that linear systems may or may not fulfill: time-invariance, causality, stability, reality.

Time-invariance. Let us define the time-shift operator $f \mapsto f_\tau := f(\cdot - \tau)$, for any admissible signal $f \in \Phi$ and any $\tau \in \mathbb{R}$. A linear system L is called *time-invariant* (or *stationary* or *translation invariant)* iff it commutes with time-shifts, that is,

$$f_\tau \in \Phi, \qquad L(f_\tau) = L(f)_\tau \qquad \forall f \in \Phi, \forall \tau > 0.^{10} \tag{1.5}$$

The systems of the examples (i)–(vi) above are time-invariant, with the exception of (iv); and so is also (vii) if P and Q have constant coefficients. [11]

Causality. A linear system L is called *causal* if, for any $f_1, f_2 \in \Phi$ and $\tau \in \mathbb{R}$,

$$f_1(t) = f_2(t) \quad \forall t < \tau \quad \Rightarrow \quad [L(f_1)](t) = [L(f_2)](t) \quad \forall t < \tau, \tag{1.6}$$

or equivalently, because of the linearity of L,

$$f(t) = 0 \quad \forall t < \tau \quad \Rightarrow \quad [L(f)](t) = 0 \quad \forall t < \tau. \tag{1.7}$$

We called signal f causal iff $f(t) = 0$ for any $t < 0$. A system is thus causal iff it maps causal signals to causal signals. We shall see that there is a mathematical reason behind the use of the term causality for both systems and signals. Just causal systems are physically realizable, so that the term *realizable* is also used as a synonymous of causal.

For instance, the systems of the examples (i)–(vii) above are causal.

Stability. Let $p \in [1, +\infty[$. We shall say that a linear system $L : \Phi \to \mathcal{D}'$ is L^p*-bounded* if it maps $\Phi \cap L^p$ to L^p and is bounded, that is,

$$\exists C_p > 0 : \forall f \in \Phi \cap L^p, \qquad \|L(f)\|_{L^p} \le C_p \|f\|_{L^p}. \tag{1.8}$$

[8] RLC circuits, rheological models, acoustic models and several other models of applied sciences and engineering correspond to the equation of the damped driven harmonic oscillator. The analogy between these models is widely exploited in engineering.

[9] In Analysis these are called *autonomous* differential equations.

[10] For any $f \in \mathcal{D}'$, $f(\cdot - \tau) = f * \delta_\tau$. This equality thus also reads $L(f * \delta_\tau) = L(f) * \delta_\tau$.

[11] Indeed, translations and linear combinations are shift-invariant. Derivatives are limits of these operations, and thus have the same property. Inversion preserves shift-invariance.

Similarly, we shall say that a linear system $L : \Phi \to \mathcal{D}'$ is C^0_b*-bounded* if it maps $\Phi \cap C^0_b$ to L^∞ and is bounded, that is, [12]

$$\exists C > 0 : \forall f \in \Phi \cap C^0_b, \qquad \|L(f)\|_{L^\infty} \le C \sup |f|. \tag{1.9}$$

(Here we assume that $f \in C^0_b$ rather than L^∞, because C^∞ is dense in C^0_b but not in L^∞.)

Finally, we shall say that a linear system $L : \Phi \to \mathcal{D}'$ is *stable* if it is C^0_b-bounded and L^p-bounded for any $p \in [1, +\infty[$. In the engineering literature a linear systems is called *BIBO stable* (BIBO = bounded input bounded output) if it is L^∞-bounded.

Further properties. Time-invariance, causality and stability are the main properties that a deterministic linear system may satisfy. However, other properties may also be of interest.

A linear system $L : \Phi \to \Psi$ is called *real* if it transforms real signals to real signals, that is, The examples (i)–(vii) above are all real systems.

A linear system is called *instantaneous* (or *memoryless)* if, at each instant t, $[L(f)])(t)$ only depends on the value of f at the same instant t. Amplifiers are the only memoryless real linear systems.

A linear system L is called *invertible* if the inverse operator L^{-1} exists. L^{-1} is linear; moreover, if L is time-invariant (causal, real, resp.), then L^{-1} is also time-invariant (causal, real, resp.).

1.3 Functional set-up

Some heuristics. Let us set $\widetilde{h}(\cdot, \tau) = L(\delta_\tau)$ for any real τ. Note that $f = f * \delta = \int_{\mathbb{R}} f(\tau)\delta_\tau \, d\tau$ for any $f \in \Phi$. [13] We may guess that

$$L(f) = L\Big(\int_{\mathbb{R}} f(\tau)\delta_\tau \, d\tau \Big) \overset{(?)}{=} \int_{\mathbb{R}} f(\tau) L(\delta_\tau) \, d\tau. \tag{1.10}$$

If L is time-invariant then $L(\delta_\tau) = L(\delta)_\tau$, whence

$$L(f) = \int_{\mathbb{R}} f(\tau) L(\delta_\tau) \, d\tau = \int_{\mathbb{R}} f(\tau) L(\delta)_\tau \, d\tau = f * L(\delta). \tag{1.11}$$

The second equality of (1.10) must be justified. Here we justify this result via a different procedure.

Lemma 1.1 *If $U : \mathcal{D} \to C^0$ is linear, sequentially continuous* [14] *and time-invariant, then there exists one and only one $T \in \mathcal{D}'$ such that*

$$U(f) = T * f \qquad \forall f \in \mathcal{D}. \tag{1.12}$$

*** Proof.** Let us set $T(\check{f}) = [U(f)](0)$. It is easy to see that T is a distribution. For any $f \in \mathcal{D}$ and $x \in \mathbb{R}$, let us set $f_x = f(\cdot - x)$ and $\check{f}(x) = f(-x)$. As U is time-invariant, we have [15]

$$[U(f)](-x) = U(f)_x(0) = [U(f_x)](0) = T[(f_x)\check{}\,] = T[(\check{f})_x] = (T * f)(-x). \tag{1.13}$$

[12] In the engineering literature a linear systems is called *BIBO stable* (BIBO = bounded input bounded output) if it is L^∞-bounded.

[13] This is not an integral, but occasionally in this chapter we shall use the integral notation because of its readability, consistently with the engineering literature. (By δ_τ we denote the Dirac mass at the point τ.)

[14] Henceforth we shall omit the specification *sequentially.*

[15] As we saw, $f_x = f * \delta_x$. We can then rewrite this formula as follows:

$$[U(f)](-x) = [U(f) * \delta_x)](0) = [U(f * \delta_x)](0) = T[(f * \delta_x)\check{}\,] = T(\check{f} * \delta_x) = (T * f)(-x).$$

For several results we shall assume that

$$\text{either} \quad \mathcal{E}' \subset \Phi \subset \mathcal{S}', \quad L : \Phi \to \mathcal{S}', \qquad \text{or} \qquad \mathcal{E}' \cap D_{\mathcal{L}} \subset \Phi \subset D_{\mathcal{L}}, \quad L : \Phi \to D_{\mathcal{L}}, \tag{1.14}$$

so that inputs and outputs of either system are Fourier transformable in the first case, Laplace transformable in the second case. Moreover, Dirac masses are admissible inputs in either case. [16]

• **Theorem 1.2** *(Convolution systems) Let $\mathcal{E}' \subset \Phi \subset \mathcal{D}'$ with continuous injections, let $L : \Phi \to \mathcal{D}'$ be linear, continuous and time-invariant, and assume that*

$$\textit{the restriction } L\big|_{\mathcal{D}} \textit{ maps } \mathcal{D} \textit{ to } C^0 \textit{ and is continuous.} \tag{1.15}$$

Then [17]

$$L(f) = L(\delta_0) * f \qquad \forall f \in \mathcal{E}', \tag{1.16}$$

$$L(\delta_0) \in L^1 \quad \Rightarrow \quad L(f) = L(\delta_0) * f \quad \forall f \in \Phi \cap \Big(\bigcup\nolimits_{p\in[1,+\infty[} L^p \cup C^0_b\Big). \tag{1.17}$$

In this case L is called a *convolution system.* Conversely it is easily checked that any convolution system is time-invariant.

* **Proof.** (i) Let $T \in \mathcal{D}'$ be as in Lemma 1.1, which we can apply because of (1.15). We claim that

$$T = L(\delta_0) \ (\in \mathcal{D}'). \tag{1.18}$$

Let $\{\rho_n\}$ be a sequence of compactly supported mollifiers, so that

$$\delta_0 * \rho_n \in C^\infty \cap \mathcal{E}' \subset \Phi, \qquad \delta_0 * \rho_n \to \delta_0 \quad \text{in } \mathcal{E}' \qquad \forall n \in \mathbb{N}.$$

As $L : \Phi \to \mathcal{D}'$ and the convolution $\mathcal{D}' \times \mathcal{E}' \to \mathcal{D}'$ are continuous, we have

$$L(\delta_0 * \rho_n) \to L(\delta_0), \qquad L(\delta_0 * \rho_n) = T * (\delta_0 * \rho_n) \to T * \delta_0 = T \qquad \text{in } \mathcal{D}'. \tag{1.19}$$

The claim (1.18) is thus established.

As $\mathcal{D}$ is (sequentially) dense in $\mathcal{E}'$, (1.12) yields (1.16) for any $f \in \mathcal{E}'$, too. (This can be proved by an approximation procedure analogous to the one that we just used.) (1.16) is thus established. If $L(\delta_0)$ is compactly supported, then we can extend this property to any $f \in \mathcal{D}'$.

(ii) By (1.16) and by the Young Theorem **??**, if $L(\delta_0) \in L^1$ then

$$\|L(\delta_0) * f\|_{L^p} \le \|L(\delta_0)\|_{L^1} \|f\|_{L^p} \qquad \forall f \in \mathcal{D}, \forall p \in [1, +\infty[, \tag{1.20}$$

$$\|L(\delta_0) * f\|_{L^\infty} \le \|L(\delta_0)\|_{L^1} \|f\|_{L^\infty} \qquad \forall f \in C^0_b. \tag{1.21}$$

As $\mathcal{D}$ is dense in L^p for any $p \in [1, +\infty[$ as well as in C^0_b, (1.17) follows. □

[16] This selection of the functional set-up is a choice of this presentation. Most of the engineering literature neither specifies the domain of definition of linear system, nor mentions functions space.
(Whenever we state an inclusion between spaces, we imply that the injection is continuous.)

[17] In the engineering literature some authors invert the implication (1.17). But this fails for amplifiers and delay systems.

Remarks 1.3 (i) As $\delta_0 \in \mathcal{E}'$, we assumed $\mathcal{E}' \subset \Phi$ so that $L(\delta_0)$ meaningful.

(ii) In (1.16) we assumed $f \in \mathcal{E}'$ since $\mathcal{E}' * \mathcal{D}' \subset \mathcal{D}'$, whereas $\mathcal{D}' * \mathcal{D}'$ is meaningless. Assuming $L(\delta_0) \in \mathcal{E}'$ would be too restrictive for applications, but the Young Theorem allows one to overcome restrictions on the support when dealing with L^p-spaces.

(iii) In the engineering literature the integrability of $L(\delta_0)$ is often assumed, and part (ii) of this theorem is often applied. Part (i) applies e.g. to amplifiers and delay systems, for which respectively $L(\delta_0) = A\delta_0$ and $L(\delta_0) = \delta_a$. □

Although some elements of $f \in \Phi \setminus \mathcal{E}'$ may not be representable as a convolution, by the next statement any output can be approximated by a sequence of convolution products.

Proposition 1.4 *Let L be as in Theorem 1.2, and $\{\rho_n\}$ be a sequence of compactly supported mollifiers. Then for any $f \in \Phi$, setting $f_n = f * \rho_n$ for any n,*

$$L(\delta_0) * f_n \to Lf \qquad \text{in } \mathcal{D}'. \tag{1.22}$$

Proof. $f_n \in \mathcal{E}'$ for any n, and $f_n \to f$ in $\mathcal{D}'$. By (1.16) and by the continuity of L, therefore $L(\delta_0) * f_n = L(f_n) \to Lf$ in $\mathcal{D}'$. □

Corollary 1.5 *Under the assumptions of Theorem 1.2, $L(\delta_0)$ characterizes the system L. More precisely, for any couple of systems $L_1, L_2 : \Phi \to \mathcal{D}'$ as in Theorem 1.2, if $L_1(\delta_0) = L_2(\delta_0)$ then $L_1 = L_2$.*

Moreover

$$\text{the system } L \text{ is causal iff the signal } L(\delta_0) \text{ is causal,} \tag{1.23}$$

$$\text{the system } L \text{ is real iff the signal } L(\delta_0) \text{ is real.} \tag{1.24}$$

Proof. The first statement stems from the foregoing Proposition.

If $f \in \mathcal{D}$ and $L(\delta_0)$ are causal, then $L(f) = f * L(\delta_0)$ is also causal. This property is then extended to any $f \in \Phi$ by the latter Proposition. The proof of the converse and of the statement on reality are obvious. □

Let us denote the unit step function by u, which is the customary notation of the Heaviside function in signal theory.

Proposition 1.6 *Let L be as in Theorem 1.2. If $u \in \Phi$ and $h := L(\delta_0) \in L^1$, then*

$$h_1(t) := [L(u)](t) = \int_{-\infty}^{t} h(r)\, dr \;\in L^\infty \qquad \forall t \in \mathbb{R}. \tag{1.25}$$

Proof. By (1.17),

$$h_1(t) = (u * h)(t) = \int_{\mathbb{R}} u(t-r)h(r)\, dr = \int_{-\infty}^{t} h(r)\, dr \qquad \forall t \in \mathbb{R}, \tag{1.26}$$

and $|h_1(t)| \le \|h\|_{L^1}$. □

By (1.26), $Dh_1 = h$. This step response thus characterizes the system, just as the impulse response h.

2 Filters and Transfer Functions

In this section we define (possibly noncausal) filters, we apply the Fourier transform, and briefly illustrate energetic aspects. We then use the Laplace transform to deal with causal signals, without assuming L^p-boundedness.

Filters. [18] The term *filter* may have two meanings. In *signal processing* (which is an engineering discipline) a filter is a physical device which modifies the signal, typically by acting on its spectral components. This process is called *filtering.* Examples include amplifiers, RLC circuits, derivators, integrators, delayed circuits, and the many systems that are constructed by composing these basic elements according to certain rules, as we shall see ahead.

In *signal analysis* (the theoretical counterpart of signal processing), a filter is a time-invariant linear and stable system. Here we call L a *stable filter* iff the following holds: [19]

(i) Φ is a topological linear space and $\mathcal{E}' \subset \Phi \subset \mathcal{S}'$ with continuous injections,
(ii) $L : \Phi \to \mathcal{S}'$ is linear, continuous and time-invariant,
(iii) the restriction $L\big|_{\mathcal{D}}$ maps $\mathcal{D}$ to C^0 and is continuous,
(iv) L is stable in the sense of (1.8) and (1.9).

By assuming $\mathcal{E}' \subset \Phi$ and (iii) we guarantee that $\delta_0 \in \Phi$, and we get (1.16), (1.17).

Examples include the linear systems (i)–(vi) of Section 1, with the exception of derivators (here $L(\delta_0) \notin L^1$) and integrators (they are not stable). Moreover, a state equation $P(D)u = Q(D)f$ represents a filter if $P(i\xi) \neq 0$ for any $\xi \in \mathbb{R}$; see Proposition **??** (Prop 5.1 of the Distribution chapter) For instance, damped harmonic oscillators, RLC circuits and rheological models are differential filters of order two.

Most typical filters select certain frequencies of the input, and annihilate (or at least strongly reduce) the others. For instance, low-pass, high-pass, band-pass filters select a range of frequencies, that are respectively either below or above a certain threshold, or confined to a prescribed band. Note that any filter that selects *real* frequencies is not causal.

A filter is characterized by the impulse responses [20]

$$h = L(\delta_0) \quad \text{in time,} \qquad \mathcal{H} = \mathcal{F}(h) \quad \text{in frequency.} \tag{2.1}$$

$\mathcal{H}$ is called the *transfer function,* or the *spectrum* of the system. [21]

Remarks 2.1 (i) Denoting by $R(\xi)$ and $iX(\xi)$ respectively the real and imaginary part of $\mathcal{H}(\xi)$, by $A(\xi)$ its modulus, and by $\varphi(\xi)$ its phase, we have

$$\mathcal{H}(\xi) = R(\xi) + iX(\xi) = A(\xi)e^{2\pi i\varphi(\xi)} \qquad \forall \xi \in \mathbb{R}. \tag{2.2}$$

[18] The theory of filters has two main actors: the filter L and the transform, $\mathcal{F}$ or $\mathcal{L}$. The transform just concerns the representation of the signal, so it is essentially ... a linguistic matter (equipped with a remarkable body of calculus, of course).

[19] This definition is customized for (possibly noncausal) signals and for the Fourier transform. Dealing with causal signals, we shall amend this definition.

[20] The notation $\mathcal{H}$ should not be confused with that of the Heaviside function, or *unit step.* In this chapter we shall denote the latter by u, consistently with the engineering literature.

[21] As we saw, the Fourier transform of a signal is called its *spectrum.* It is natural to use the same term for the system and for signals: indeed (see Theorem 2.2 ahead) the role of the system is somehow comparable to that of the input signal.

The denomination *transfer function* is most often used for the Laplace transform, see ahead. In the theory of differential equations, the impulse response is called the *fundamental* or *elementary solution* of the equation.

In Signal Processing the functions A and φ are respectively called the *amplitude spectrum* and the *phase spectrum* of the system. $A(\xi)$ and $\phi(\xi)$ are also called the *gain* and the *phase shift* of the system at the frequency ξ.

(ii) By (1.26) and (2.1),

$$\mathcal{H}(0) = \int_{\mathbb{R}} h(t)\, dt = \lim_{t\to+\infty} h_1(t). \tag{2.3}$$

$|\mathcal{H}(0)|$ represents how much the system asymptotically amplifies the unit step, and is called the *gain of the system* tout court. This is quite relevant for applications, since in electronic circuits the transient is typically quick and of minor interest. □

The next result is at the basis of the use of operational calculus in Signal Analysis. Here we state it for stable filters using the Fourier transform, and ahead for causal filters using the Laplace transform.

• **Theorem 2.2** *(Fundamental formula of stable filters) Let* $L : \Phi \to \mathcal{S}'$ *be a filter, let* $f \in \Phi \cap L^p$ $(p \in [1,2])$*, and define the Fourier transformed functions* $F := \mathcal{F}(f)$, $G := \mathcal{F}(L(f))$, $\mathcal{H} := \mathcal{F}(L(\delta_0))$*. Then*

$$G(\xi) = \mathcal{H}(\xi)\, F(\xi) \qquad q\forall\xi \in \mathbb{R}, \text{ and in } L^{p'}. \tag{2.4}$$

Proof. $L(f) \in L^p$ by the axiom (iv) of filters. Hence $F, G \in L^{p'}$, by the Hausdorff-Young Theorem **??**. [22]

For $p = 1$, (2.4) follows from the Convolution Theorem of [Fourier section]. This is then extended by continuity to L^p spaces, since $L^p \cap L^1$ is dense in L^p. □

Corollary 2.3 *If* $L : \Phi \to \mathcal{S}'$ *is a filter, then:* [23]

(i) For any $p \in [1,2]$,

$$\mathcal{F}L : \Phi \cap L^p \to L^{p'} : f \mapsto \mathcal{F}(L(f)) \text{ is linear and continuous.} \tag{2.5}$$

Moreover,

$$\|\mathcal{F}(L(f))\|_{L^{p'}(\mathbb{R})} \le \|\mathcal{H}\|_{L^\infty(\mathbb{R})} \|f\|_{L^p(\mathbb{R})} \qquad \forall f \in \Phi \cap L^p. \tag{2.6}$$

The norm of the operator $\mathcal{F}L$ *is thus bounded by* $\|\mathcal{H}\|_{L^\infty(\mathbb{R})}$ $(\le \|h\|_{L^1(\mathbb{R})})$.

(ii) Assuming that $L^2 \subset \Phi$ *and omitting restrictions,*

$$\widehat{L} := \mathcal{F}L\mathcal{F}^{-1} : L^2_t \to L^2_t \text{ is linear and continuous.} \tag{2.7}$$

Moreover, $\widehat{L}$ *is multiplicative:*

$$[\widehat{L}(F)](\xi) = \mathcal{H}(\xi)\, F(\xi) \qquad q\forall\xi \in \mathbb{R}, \forall F \in L^2_\xi. \tag{2.8}$$

The norm of the operator $\widehat{L}$ *is thus* $\|\mathcal{H}\|_{L^\infty(\mathbb{R})}$ $(\le \|h\|_{L^1(\mathbb{R})})$.

(We can state part (ii) just for $p = 2$, since for the other p $\mathcal{F}$ is not surjective.)

[22] As usual, with $p' = p/(p-1)$ if $p \ne 1$ and $\infty' = 1$.

[23] One should not confuse L (the customary notation for linear systems) with the Laplace transform $\mathcal{L}$ or the Lebesgue space L^p.

Remarks 2.4 (i) By Theorem 2.2, the system acts independently on each spectral component of the input signal. In this sense, the Fourier transform is a *spectral resolution* (i.e., a sort of *diagonalization)* of any filter. [24]

(ii) Because of the symmetry between signals and filter of (2.4) and of other results, in the engineering literature one often deals with filters and signals on the same footing. So for instance terms like spectrum, energy gain, causal and others are used for both signals and systems.

(iii) For any $\gamma \in \mathcal{F}(L^1)$ $(\subset C_b^0)$, $h = \widetilde{\mathcal{F}}(\gamma)$ is the impulse response of a filter, and this has the spectrum $\mathcal{H} = \mathcal{F}(h) = \gamma$.

It is not easy to characterize $\mathcal{F}(L^1)$, but in practice this does not prevent engineers from designing and then efficiently producing electronic filters for a desired shape of their spectrum. □

Energy of signals. Let us consider an electrical current flowing along a conducting wire having unit resistance. If $f(t)$ is the current density at any instant $t \in \mathbb{R}$ and $f \in L^2$, then $\|f\|_{L^2}^2$ coincides with the total energy of the signal (namely, the energy content of that current). For any real a, b with $a < b$, $\int_a^b |f(t)|^2 \, dt$ is thus the energy of the signal in the time interval $]a, b[$. Signals of L^2 are then named *energy signals.* Accordingly, $|f(t)|^2$ is regarded as the time-density of energy at the instant t.

As by the Plancherel Theorem

$$\int_{\mathbb{R}} |F(\xi)|^2 \, d\xi = \int_{\mathbb{R}} |f(t)|^2 \, dt \qquad (F := \mathcal{F}(f)), \tag{2.9}$$

the energy is similarly partitioned with respect to frequency. $|F(\xi)|^2$ is thus interpreted as the frequency-density of energy at the frequency ξ, and the function $\xi \mapsto |F(\xi)|^2$ is called the *energy spectrum* of the signal.

Under the assumptions of Theorem 2.2 for $p = 2$, the system transforms the energy spectrum of the signal as follows:

$$|G(\xi)|^2 = |\mathcal{H}(\xi)|^2 \, |F(\xi)|^2 \qquad q\forall \xi \in \mathbb{R}. \tag{2.10}$$

The function $|\mathcal{H}|^2$ is then called the *power spectrum* of the system.

Thus the system acts independently on each spectral component of the energy. In this sense, the Fourier transform *diagonalizes* the action of any filter not only on the signals but also on the energy.

Laplace transform of causal filters. So far in this section we considered stable filters and applied the Fourier transform. Dealing with causal signals, we define L a *causal filter* iff the following holds:

(i) Φ is a topological linear space and $\mathcal{E}' \cap D_{\mathcal{L}} \subset \Phi \subset D_{\mathcal{L}}$ with continuous injections,

(ii) $L : \Phi \to D_{\mathcal{L}}$ is linear, continuous, time-invariant and causal,

(iii) the restriction of L maps $\Phi \cap \mathcal{D}$ to C^0 continuously.

Note that then $\delta_0 \in \Phi$ and $h := L(\delta_0) \in D_{\mathcal{L}}$. Let us denote its Laplace transform by $\widetilde{\mathcal{H}}$: [25]

$$\widetilde{\mathcal{H}}(s) := [\mathcal{L}(h)](s) = \int_{\mathbb{R}} h(t) e^{-st} \, dt \qquad \forall s \in \mathbb{C}_{\lambda(h)}. \tag{2.11}$$

[24] This actually holds for any linear system. Here time-invariance enters just in expressing the continuity constant in terms of the transform of $h = L(\delta_0)$.

[25] This integral should be replaced by a duality pairing: in section [Laplace transform of distributions] we formulated this definition more precisely.

We know that this function is holomorphic in the half-plane of convergence $\mathbb{C}_{\lambda(h)}$. As we did for the Fourier transform, here also we call $\widetilde{\mathcal{H}}$ the *transfer function.* As we know, if $h := L(\delta_0) \in L^1$, $\widetilde{\mathcal{H}}$ extends the function $\mathcal{H}$ of (2.1); more precisely,

$$\widetilde{\mathcal{H}}(i\xi) = [\mathcal{L}(h)](i\xi) = \int_{\mathbb{R}} h(t)e^{-i\xi t}\,dt = [\mathcal{F}(h)](\xi/(2\pi)) = \mathcal{H}(\xi/(2\pi)) \qquad \forall \xi \in \mathbb{R}. \tag{2.12}$$

Theorem 2.5 *(Fundamental formula of causal filters) Let $L : \Phi \to D_{\mathcal{L}}$ be a causal filter, let $f \in \Phi$, and define the Laplace transformed functions $F := \mathcal{L}(f)$, $G := \mathcal{L}(L(f))$, $\widetilde{\mathcal{H}} := \mathcal{L}(L(\delta_0))$. Then*

$$\lambda(L(f)) \le \max\{\lambda(L(\delta_0)), \lambda(f)\}, \qquad G(s) = \widetilde{\mathcal{H}}(s)\,F(s) \qquad \forall s \in \mathbb{C}_{\lambda(L(f))}. \tag{2.13}$$

This obviously entails

$$|G(s)|^2 = |\widetilde{\mathcal{H}}(s)|^2\,|F(s)|^2 \qquad \forall s \in \mathbb{C}_{\lambda(L(f))}. \tag{2.14}$$

Thus function $|\widetilde{\mathcal{H}}|^2$ is the power spectrum of the causal system.

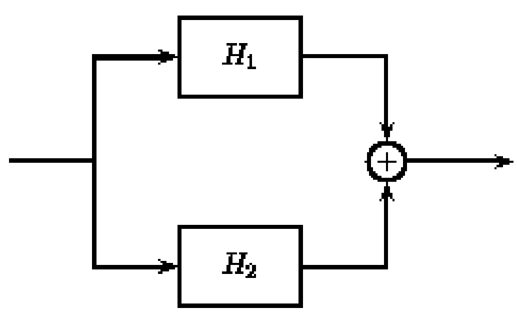
3 Filter Composition

Filters can be arranged in several ways. In this section we deal with the basic connections in parallel, in series, and in *retroaction* (or *feedback)* of two filters L_1 and L_2, derive the state equation and the transfer function of each of these composed systems.

We also briefly address the stability of this model under variations of the transfer functions of the components of the system. This is relevant for applications, since real device always exhibit imperfections. [26]

Here first we deal with stable filters, and use the Fourier transform; afterwards we outline the modifications that are needed to apply the Laplace transform to casual filters.

(i) Parallel connection. The composed system L is represented by the following *block diagram,* which represents the signal path. The bifurcation of the path does not represent any partition of this flow of information. For instance this applies if the input signal represents a tension rather than a current density (which are intensive and extensive quantities, respectively). Consistently with the literature, in the diagram we represent the filters by their transfer function in frequency. Note the presence of the *adder,* which simply sums the inputs.



[26] * Above we referred to stability with respect to variations of signal. This corresponds to the following question: does a small variation of the input (e.g., a noise) entail a small variation of the output? One may also consider a different type of stability, in which the system itself is varied, or rather its transfer function: in a composed system, does a small variation of the $\mathcal{H}_i$'s entail a small variation of the overall transfer function $\mathcal{H}$?

Here $\Phi_1 = \Phi_2 =: \Phi$, and the system L is defined by the map [27]

$$L : f \mapsto g = L_1(f) + L_2(f) \qquad \forall f \in \Phi. \tag{3.1}$$

For any $p \in [1,2]$ and any $f \in \Phi \cap L^p$, setting $F := \mathcal{F}(f)$, $G := \mathcal{F}(L(f))$, $\mathcal{H}_i := \mathcal{F}(h_i)$, we have

$$g(t) = [(h_1 + h_2) * f](t) \qquad q\forall t \in \mathbb{R}, \tag{3.2}$$

$$G(\xi) = [\mathcal{H}_1(\xi) + \mathcal{H}_2(\xi)]F(\xi) \qquad q\forall \xi \in \mathbb{R}. \tag{3.3}$$

Proposition 3.1 *(Parallel connection) For $i = 1, 2$, let $L_i : \Phi \to \mathcal{S}'$ be a filter with $h_i = L_i(\delta_i) \in L^1$. Then $L : f \mapsto g = L_1(f) + L_2(f)$ is a filter with impulse response*

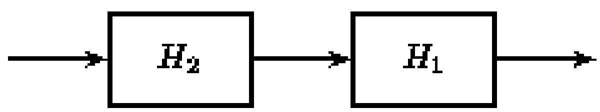
$$h = h_1 + h_2 \quad \textit{in time}, \qquad \mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2 \quad \textit{in frequency.} \tag{3.4}$$

Small pointwise variations of h_1 and h_2 entail a small pointwise variation of h; the same holds in the norm of L^p for any $p \in [1, +\infty]$. This also applies to $\mathcal{H}$ pointwise and in L^p, for any $p \in [2, +\infty]$.

(Here p is confined to $[2, +\infty]$ because of the Hausdorff-Young Theorem.)

Remark 3.2 The same result applies to causal filters $L_i : \Phi \to D_{\mathcal{L}}$ $(i = 1, 2)$, using the Laplace transform. □

(ii) Series (or cascade or tandem) connection. Let us assume that $L_1(\Phi_1) \subset \Phi_2$. The composed system L is represented by the following block diagram.



This is defined by the map [28]

$$L : f \mapsto g = L_2(L_1(f)) \qquad \forall f \in \Phi_1. \tag{3.5}$$

For any $p \in [1,2]$ and any $f \in \Phi_1 \cap L^p$, setting $F := \mathcal{F}(f)$, $G := \mathcal{F}(L(f))$, $\mathcal{H}_i := \mathcal{F}(h_i)$ $(i = 1, 2)$, we have

$$g(t) = [h_2 * h_1 * f](t) \qquad q\forall t \in \mathbb{R}, \tag{3.6}$$

$$G(\xi) = \mathcal{H}_2(\xi)\, \mathcal{H}_1(\xi)\, F(\xi) \qquad q\forall \xi \in \mathbb{R}. \tag{3.7}$$

(We can write $h_1 * h_2 * f$ since in L_1 the convolution is associative.)

Proposition 3.3 *(Series connection) Let $L_1 : \Phi_1 \to \Phi_2$ and $L_2 : \Phi_2 \to \mathcal{S}'$ be filters with $h_i := L_i(\delta_0) \in L^1$ for $i = 1, 2$. Then $L : f \mapsto g = L_2(L_1(f))$ is a filter with impulse response*

$$h = h_2 * h_1 \quad \textit{in time}, \qquad \mathcal{H} = \mathcal{H}_2\, \mathcal{H}_1 \quad \textit{in frequency.} \tag{3.8}$$

Small variations of h_1 and h_2 in L^1 entail a small variation of h in L^1. The same applies to $\mathcal{H}$ in C^0_b, and also for any $\xi \in \mathbb{R}$.

[27] This formula applies also if L_1 and L_2 are nonlinear, or time-varying systems, or unstable systems.
[28] This formula applies also if L_1 and L_2 are nonlinear, or time-varying systems, or unstable systems.

Proof. By the Young Theorem, $h = h_2 * h_1 \in L^1$, hence L is a filter. By applying the Fourier transform, the second equality of (3.8) follows pointwise in C_b^0.

Let us denote by Δh_i a variation of h_i $(i = 1, 2)$, and by Δh the corresponding variation of $h = h_2 * h_1$; that is,

$$\Delta h = (h_2 + \Delta h_2) * (h_1 + \Delta h_1) - h_2 * h_1.$$

By the Young Theorem,

$$\begin{aligned} \|\Delta h\|_{L^1} &\leq \|\Delta h_2 * h_1\|_{L^1} + \|h_2 * \Delta h_1\|_{L^1} + \|\Delta h_2 * \Delta h_1\| \\ &\leq \|\Delta h_2\|_{L^1} \|h_1\|_{L^1} + \|h_2\|_{L^1} \|\Delta h_1\|_{L^1} + \|\Delta h_2\|_{L^1} \|\Delta h_1\|_{L^1}. \end{aligned} \tag{3.9}$$

Thus $\|\Delta h\|_{L^1}$ depends on $\|\Delta h_1\|_{L^1}$ and $\|\Delta h_2\|_{L^1}$ with continuity.

The statement concerning $\mathcal{H}$ is obvious. □

Remarks 3.4 (i) An analogous result holds if $L_1 : \Phi_1 \to \Phi_2 \subset D_{\mathcal{L}}$ and $L_2 : \Phi_2 \to D_{\mathcal{L}}$ are assumed to be causal filters. In this case the Laplace transform is used, and no reference is made to L^p-spaces. Details are left to the Reader.

(ii) Some further properties can be proved. For instance, if

$$p, q \in [1, 2], \quad r \in [1, +\infty], \quad p^{-1} + q^{-1} = 1 + r^{-1}, \quad h_1 \in L^1 \cap L^p, h_2 \in L^1 \cap L^q, \tag{3.10}$$

then

$$\mathcal{H}_1 \in C_b^0 \cap L^{p'}, \qquad \mathcal{H}_2 \in C_b^0 \cap L^{q'}, \tag{3.11}$$

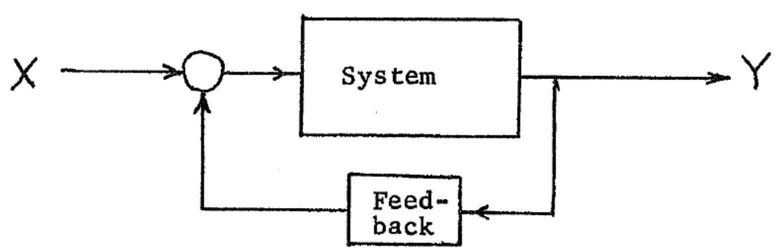
$$h = h_2 * h_1 \in L^r, \qquad \|h\|_{L^r} \leq \|h_2\|_{L^p} \|h_1\|_{L^q}. \tag{3.12}$$

Moreover, $(p')^{-1} + (q')^{-1} = (r')^{-1}$ by the second inequality of (3.10). By (3.9) and (3.11), then

$$\mathcal{H} \in C_b^0 \cap L^{r'}, \qquad \|\mathcal{H}\|_{L^{r'}} \leq \|\mathcal{H}_1\|_{L^{p'}} \|\mathcal{H}_2\|_{L^{q'}}. \tag{3.13}$$

Corresponding results for variations of h_1, h_2 can be derived as above. □

(iii) Feedback connection. Let us assume that L_1, L_2 are two filters as above, with $L_1(\Phi_1) \subset \Phi_2$ and $L_2(\Phi_2) \subset \Phi_1$. The feedback connection L is represented by the following block diagram.



As we pointed out for parallel combinations, the bifurcation of the path does not represent any partition of this flow of information.

We shall derive the transfer function of the composed system, and shall compare its action with that of the forward element L_1 alone.

The engineering literature distinguishes between negative and positive feedback. In either system we have the following signals in time:

f: the input of the composed system,

g: the output of the *feedforward element* L_1, which coincides with the input of the *feedback element* L_2, and also coincides with the output of the composite system,

ℓ: the output of the *feedback element* L_2; this and f are the two inputs of the combiner element (either an adder or a subtractor);

m: the output of the combiner element, which is the input of L_1.

The feedback connection corresponds to the following equations:

$$m = f \pm \ell, \qquad g = L_1(m), \qquad \ell = L_2(g), \tag{3.14}$$

with the plus sign for negative feedback, the minus for positive feedback. By eliminating ℓ and m, we get a single equation:

$$\begin{aligned} g &= [L_1(f - L_2(g))] \quad \text{for negative feedback,} \\ g &= [L_1(f + L_2(g))] \quad \text{for positive feedback.} \end{aligned} \tag{3.15}$$

The simple ODE $y' + ay = f$ can be represent as the state equation of a feedback system.

* **Remarks 3.5** (i) Via Fourier transform, the signals f, g, ℓ, m may also be represented as functions of frequency: $\widehat{f}, \widehat{g}, \widehat{\ell}, \widehat{m}$, resp.. Setting $\bar{L}_i = \mathcal{F} L_i \mathcal{F}^{-1}$ $(i = 1, 2)$, the equations (3.14) and (3.15) yield

$$\begin{aligned} &\widehat{m} = \widehat{f} \pm \widehat{\ell}, \qquad \widehat{g} = \bar{L}_1(\widehat{m}), \qquad \widehat{\ell} = \bar{L}_2(\widehat{g}), \\ &\widehat{g} = [\bar{L}_1(\widehat{f} \mp \bar{L}_2(\widehat{g}))]. \end{aligned} \tag{3.16}$$

Thus the system equations do not depend on the form of representation of the signals. Actually, the input-output relations concern signals rather than functions, as it is well known by engineers.

(ii) The formulation (3.16) does not depend on the linearity of the operators L_1 and L_2, but just on the linearity of the operations involved in the composition of these elements (here the feedback arrangement). Therefore (3.16) also applies to nonlinear operators. However, our analysis will rest on linearity, which entails the multiplicative representation of systems in frequency, see Theorem 2.2. □

Negative feedback. Because of the stability of the filters, for any $p \in [1, +\infty[$ and any $f \in \Phi_1 \cap L^p$,

$$g \in \Phi_2 \cap L^p, \qquad \ell, m \in \Phi_1 \cap L^p. \tag{3.17}$$

As $L_1(m) = m * h_1$ and $L_2(g) = g * h_2$, $(3.15)_1$ yields

$$g(t) = [h_1 * (f - h_2 * g)](t) \qquad q\forall t \in \mathbb{R}. \tag{3.18}$$

By applying the Fourier transform and setting $F := \mathcal{F}(f)$, $G := \mathcal{F}(L(f))$, $\mathcal{H}_i := \mathcal{F}(h_i)$ $(i = 1, 2)$, we get the algebraic equation

$$G(\xi) = \mathcal{H}_1(\xi)[F(\xi) - \mathcal{H}_2(\xi)G(\xi)] \qquad q\forall \xi \in \mathbb{R}, \tag{3.19}$$

that is, provided that the denominator does not vanish,

$$G(\xi) = \frac{\mathcal{H}_1(\xi)}{1 + \mathcal{H}_1(\xi)\,\mathcal{H}_2(\xi)} F(\xi) \qquad q\forall \xi \in \mathbb{R}. \tag{3.20}$$

Thus L is a filter with impulse response

$$\mathcal{H}(\xi) = \frac{\mathcal{H}_1(\xi)}{1 + \mathcal{H}_1(\xi)\,\mathcal{H}_2(\xi)} \qquad \forall \xi \in \mathbb{R} \text{ (in frequency)}, \tag{3.21}$$

$$h(t) = \mathcal{F}^{-1}\left(\frac{\mathcal{H}_1(\xi)}{1 + \mathcal{H}_1(\xi)\,\mathcal{H}_2(\xi)}\right)(t) \qquad q\forall t \in \mathbb{R} \text{ (in time)}. \tag{3.22}$$

Next we assume that

$$\mathcal{H}_1(\xi), \mathcal{H}_2(\xi) \in \mathbb{R}, \qquad \mathcal{H}_1(\xi), \mathcal{H}_2(\xi) \geq 0 \qquad \forall \xi \in \mathbb{R}, \tag{3.23}$$

so that the filters L_1 and L_2 modify each spectral amplitude leaving the phase unchanged. This condition excludes delays in these two filters, since a delay in the impulse response shifts the phase of the spectrum, cf. (**??**). [a formula of Fourier chapter]

Proposition 3.6 *(Negative feedback) Let* $L_1 : \Phi_1 \to \Phi_2$ *and* $L_2 : \Phi_2 \to \Phi_1$ *be filters with* $h_i := L_i(\delta_0) \in L^1$ *for* $i = 1, 2$*, and let* $\mathcal{H}_1, \mathcal{H}_2 \geq 0$ *pointwise in* $\mathbb{R}$*. Then the negative feedback operator* L *is a filter with impulse responses* (3.21) *and* (3.22)*. Moreover,*

$$(0 \leq)\ \mathcal{H}(\xi) \leq \mathcal{H}_1(\xi) \qquad \forall \xi \in \mathbb{R}, \tag{3.24}$$

$$\|\mathcal{H}\|_{L^2} \leq \|\mathcal{H}_1\|_{L^2}, \qquad \|h\|_{L^2} \leq \|h_1\|_{L^2}. \tag{3.25}$$

Proof. (3.24) stems from (3.21) and (3.23). This entails the first inequality of (3.25). The second one follows from the Plancherel Theorem. □

Remarks 3.7 (i) The same result holds if $L_1 : \Phi_1 \to \Phi_2$ and $L_2 : \Phi_2 \to \Phi_1$ are causal filters, with $\Phi_1, \Phi_2 \subset D_{\mathcal{L}}$. In this case the Laplace transform is applied, and L^p-spaces are not used. Details are left to the Reader.

(ii) By (3.25), for any frequency the gain of the composed system is smaller than that of L_1. So the presence of L_2 reduces the effect of L_1, and has a stabilizing effect, as we already noticed. For this reason, negative feedback are widely used in electronic devices, for instance in servomechanisms. For the impulse response apparently the analogous comparison fails. [29] □

*** Reversed systems.** From a mathematical point of view, systems may also act backward; namely, the flow of information can be reversed. By Theorem 2.2, the inverse operator L_i^{-1} corresponds to the transfer function $1/\mathcal{H}_i$ for $i = 1, 2$, assuming that $\mathcal{H}_i \neq 0$ for any real ξ.

The inverse of the feed-back system is defined by the inverse operator $L^{-1} : g \mapsto f$. Here the signal σ that is produced by L_2^{-1} merges the output of L_1^{-1} rather than its input: σ is "fed forward" rather than backward. Indeed L^{-1} is what is called a *feed-forward* system.

More specifically, let us consider *negative* feedback. A simple computation shows that (3.21) is equivalent to the following relation between inverse transfer functions:

$$\mathcal{H}(\xi)^{-1} = \mathcal{H}_1(\xi)^{-1} + \mathcal{H}_2(\xi)^{-1} \qquad \forall \xi \in \mathbb{R}. \tag{3.26}$$

The feed-forward system L^{-1} thus coincides with the parallel arrangement of the inverse systems L_1^{-1} and L_2^{-1}. Note that *working in reverse* the subtractor has become and adder.

Finally, notice that the inverse of a series combination $L_2 \circ L_1$ coincides with the series arrangement of the inverses: $L_1^{-1} \circ L_2^{-1}$.

[29] (It may be problematic to state a negative property...)