Fourier Series and Music Theory — DRAFT —

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Music is the sound of mathematics [1]

Abstract. These notes introduce some basic elements of music theory using the mathematical language, in particular algebraic relations, constructions related to Fourier theory, mathematical-physical issues related to musical instruments. Attention is devoted to the theory of tuning, in particular to Pythagorean tuning, Ptolemaic tuning, and equal temperament.

Contents.

1 Basic Elements

Music affects our feelings and our mood. It has a strong impact on the cultural, social and economical life of our society: music moves the world. [2]

Music is a highly interdisciplinary topic: it involves acoustics, audiology, auditorial physiology, psychoacoustics (thus acoustics and psychology), cognitive science, mechanics, electronics, informatics, engineering, and so on. Mathematics provides language, structure and calculus to science and engineering, and pervades many of the above disciplines. In particular mathematics is at the basis of the theory of sound and of the construction of musical scales.

[1] Leibniz wrote: "Musica est exercitium arithmeticae occultum nescientis se numerare animi." (Music is a hidden arithmetic exercise of the soul, which does not know that it is counting.)

[2] In the Middle Age, music was regarded as one of the seven *liberal arts.* Arithmetic, geometry, music and astronomy formed the *quadrivium.* Jointly with grammar, logic and rethoric (the *trivium),* this constituted the basis of the education, and allowed access to the higher university studies of philosophy and theology.

In these notes we shall outline some elements of music theory, emphasizing mathematical aspects. We shall see that this also involves arithmetic, Fourier analysis, algebraic structures and mathematica physics.

1.1 The musical signal

The musical signal is characterized by:

— *Frequency.* Usually, one singles out the frequency of the fundamental component of a sound. This is the greatest common divisor of the frequencies of its partial tones. [3]

— *Duration.* This is often expressed in terms of relative duration; for instance, the musical score indicates the ratio between the duration of a tone and that of the other tones.

— *Intensity.* The acoustic power carried by the acoustic wave. For each pure tones, this is proportional to the square of the wave amplitude.

— *Timbre* (also called the *color of the sound).* This reflects how the acoustic power is distributed among pure tones. It depends on the source of the sound.

These acoustic elements are combined with musical features such as melody, harmony, tonality, rhythm, meter, texture and form.

Music occurs in so many styles that it is not easy to establish general criteria to distinguish it from other acoustic experiences. (Would devotees of classical music regard *hard metal rock* as music, or vice versa?) Anyway, we can assume that the musical sound consists of overlapping atomic *acoustic events,* that are periodic (or at least close to being so) on a small time-scale, at variance with noise.

These basic musical elements are *tones* (or *notes).* To be precise, the former term denotes a sound characterized by precise frequency and duration, the second one rather denotes the corresponding symbol on the *staff* (i.e., the musical sheet). However, the difference between these two meanings may be subtle, and we shall come back to this issue ahead. From a less linguistic and more musical point of view, let us define *pure tone* a sinusoidal wave, that is, the real or imaginary part of a function like Ae^{irt} with $A > 0$ and $r \in \mathbf{R}$. [4] The sound that is produced either by the human voice or by a musical instrument is a superposition of pure tones. However a real sound may also include components that are *inharmonic,* see ahead.

Fourier series. As we know, any sufficiently regular periodic function $u = u(t)$ of period T can be represented as a Fourier series of harmonics: [5]

$$\begin{aligned} &u(t) = \sum_{k=-\infty}^{+\infty} c_k e^{ik2\pi t/T} = \frac{a_0}{2} + \sum_{k=1}^{\infty} \big[a_k \cos(k2\pi t/T) + b_k \sin(k2\pi t/T)\big], \\ &c_k \in \mathbf{C}, \quad a_k = \frac{c_k + c_{-k}}{2}, \quad b_k = i\frac{c_k - c_{-k}}{2} \qquad \forall k \in \mathbf{N}. \end{aligned} \tag{1.1}$$

We assume that $u(t)$ is real, so that a_k and b_k are also real for any k. The first harmonic is called the *fundamental tone;* let us denote its frequency by f_1. The other components are called *overtones,* and have frequencies $n/T = nf_1$, for all integer n. (So the first overtone is the second harmonic, and so on.) The frequency is usually measured in Hertz: 1 Hz = 1 cycle/second.

Acoustics and psychoacoustics. [6] Music is produced either by the human voice or by a musical instrument. In either case a vibrator produces a variation of pressure $u = u(t)$, and a resonator

[3] Ahead we shall mention the surprising phenomenon known as *missing fundamental.*

[4] In passing, note that in mathematical analysis any function $u : \mathbf{R}^N \to \mathbf{C}$ is called *harmonic* if $\Delta u = \sum_{k=1}^{N} D_{x_i}^2 u = 0$. As Ae^{irt} is an eigenfunction of D^2 for $N = 1$, why is it called *harmonic?* The point is that the function $f : \mathbf{C} \to \mathbf{C} : z \mapsto Ae^{irz}$ is holomorphic, hence harmonic: $\Delta f(x + iy) = 0$. (*Harmonic analysis,* is a branch of mathematical analysis and includes the Fourier theory.)

[5] Regularity is a basic concern of mathematicians. Here we shall always assume that the necessary regularity conditions are fulfilled.

[6] Psychoacoustics should not be confused with the *cognitive science* of music.

amplifies the components of certain frequencies. If this signal is periodic, then it can be represented as a Fourier series, namely as a series of *harmonics,* see (1.1). [7] The k-th harmonic has frequency $f_k = kf_1$ and amplitude

$$A_k = \sqrt{a_k^2 + b_k^2} = \sqrt{|c_k|^2 + |c_{-k}|^2}/\sqrt{2}.$$

The *acoustic power* associated to the frequency f_k is proportional to A_k^2 (let us denote by σ the proportionality factor). In other terms, denoting by $E_k(\varepsilon)$ the energy conveyed in a time interval of length ε, $\lim_{\varepsilon\to 0} E_k(\varepsilon)/\varepsilon = \sigma A_k^2$. Therefore, as the period T of acoustic waves is very small, the total acoustic power of the signal $u(t)$ can be approximated by

$$\frac{1}{T}\int_{-T/2}^{T/2} |u(t)|^2\, dt \simeq \sigma \sum_{k=0}^{\infty} A_k^2 = \sigma \sum_{k=0}^{\infty} (a_k^2 + b_k^2) = \frac{\sigma}{2} \sum_{k=0}^{\infty} (|c_k|^2 + |c_{-k}|^2). \tag{1.2}$$

If the sound includes *inharmonic* (that is, non T-periodic) components, of course they also contribute to the total acoustic power.

One defines *acoustic intensity* (or sound intensity) the ratio between the acoustic power and the area on which this power is distributed. The related *acoustic intensity level* is usually expressed using the *decibel* (dB) as unit of measure. The number of decibels of a sound, I_{dB}, is 10 times the logarithm in basis 10 of the ratio between its actual intensity, I, and a reference intensity, I_0:

$$I_{dB} = 10\ \log_{10}(I/I_0), \qquad \text{i.e.,} \qquad I = I_0\ 10^{I_{dB}/10}. \tag{1.3}$$

Usually I_0 is conventionally fixed at $10^{-12} W/m^2$.

Note that the representation in decibels is not used just for sounds: it refers to the logarithm rescaling, and is used to transform other quantities, too.

Loudness. The acoustic intensity and the acoustic intensity level are physical quantities. The related notion of *loudness* is a perceptual entity and concerns psychoacoustics. So the acoustic intensity level is objective, loudness is its subjective counterpart. The latter can be evaluated by getting responses from human observers, rather than by physical measurements.

Often loudness is related to the *acoustic intensity* I on the basis of the Fechner law

Fechner law: a sensation is proportional to the logarithm of the stimulus. (1.4)

Here it is assumed that the stimulus is a physical physical, that is, it can be expressed in a precise quantitive way and can be measured. On the other hand, it is not clear how a sensation might be measured in such an objective way. A sensation is rather a psychophysical entity, which can detected via psychophysical methoda (essentially interviews). Thus (1.4) is a psychophysical law. (This requires fixing the proportionality factor, but this depends on the units of measure. Here the emphasis is on the logarithmic dependence.)

This suggests to define the loudness via a formula similar to (1.3):

$$\widetilde{I}_{dB} = 10\ \log_{10}(I/\widetilde{I}_0) \qquad \text{whence} \qquad \widetilde{I}_{dB} = 0\ \text{ if }\ I = \widetilde{I}_0. \tag{1.5}$$

(We still denote the index dB because of the logarithmic scaling.) Above we conventionally fixed I_0 at $10^{-12} W/m^2$, and this yielded the definition of the acoustic intensity level I_{dB}. Here instead we define $\widetilde{I}_0$ as the threshold of minimal perceptibility; this can be determined through interviews, so that (1.5) becomes a psychophysical definition of loudness.

The difference between the (1.3) and (1.5) is especially clear if one considers dependence on frequency. In order to simplify our discussion let us assume that the sound source is a pure tone

[7] A *harmonic* is a sinusoidal function (namely, an exponential with imaginary exponent, thus a cosine or a sine is we consider the real or imaginary part), with frequency proportional to an integer multiple of the fundamental frequency. So is $Ae^{ik2\pi t/T}$ for any $A > 0$ and any $k \in \mathbf{N}$.

of frequency f. We assumed $I_0 = 10^{-12} W/m^2$ independently of f, hence the acoustic intensity level does not depend on f. Our perception instead is highly frequency dependent, see Figure 1. At about 2 kHz our perception is optimal, and the thresholds $\widetilde{I_0}$ is not far from the conventional value $I_0 = 10^{-12} W/m^2$. But the more the sound departs from that frequency, the weaker is our perception. In particular, the lower is the intensity of a sound, the less we perceive its components of low frequency. Outside the range of about 20–20.000 Hz we do not perceive any sound.

In this context the pain (and auditory damage) threshold is set at about 120–130 decibel (= $1\ W/m^2$). Thus the span between minimal perceptibility and pain is about 12 orders of magnitude. [8]

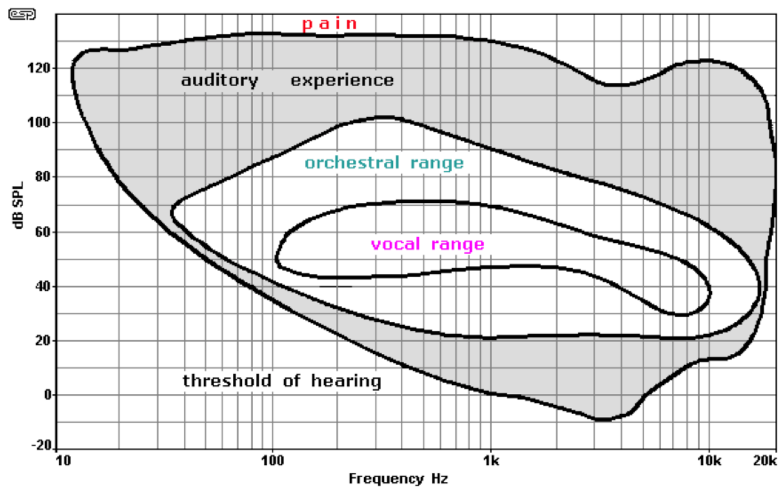

Figura 1: Hearing-range, represented in the plane of frequency and acoustic intensity level.

Beats. Beats are an acoustic phenomenon that is due to the superposition of two waves of different frequencies and the same amplitude, for instance

$$u(t) = A[\sin(\nu_1 t + \varphi_1) + \sin(\nu_2 t + \varphi_2)], \tag{1.6}$$

for some $\nu_1, \nu_2 \in \mathbf{Z} (\nu_1 \neq \nu_2)$ and $\varphi_1, \varphi_2 \in \mathbf{R}$. Let us recall the classical *prosthaphaeresis formula*

$$\sin x + \sin y = 2 \sin \frac{x+y}{2} \cos \frac{x-y}{2} \qquad \forall x, y \in \mathbf{R}, \tag{1.7}$$

and take $x = \nu_1 t + \varphi_1$ and $y = \nu_2 t + \varphi_2$. By setting

$$\nu = \frac{\nu_1 + \nu_2}{2}, \quad \delta\nu = \frac{\nu_1 - \nu_2}{2}, \quad \varphi = \frac{\varphi_1 + \varphi_2}{2}, \quad \delta\varphi = \frac{\varphi_1 - \varphi_2}{2},$$

we get

$$u(t) = A \sin(\nu t + \varphi) \cos(\delta\nu\, t + \delta\varphi). \tag{1.8}$$

The additive superposition of two waves of the same amplitude and of frequencies $\nu \pm \delta\nu$ thus produces the multiplicative superposition of two waves of frequencies ν and $\delta\nu$. If $\delta\nu \ll \nu$ (i.e., $\nu_1 \simeq \nu_2$), we thus get oscillations of frequency ν *modulated* by oscillations of frequency $\delta\nu$, with modulation period $1/\delta\nu \gg 1/\nu$. [9] Graphically this looks like a periodic wave whose amplitude has low frequency oscillations, see Figure 2.

[8] The acoustic intensity level of certain rock concerts may attain 130–140 dB, for the enjoyment of the public.

[9] In telecommunications this is used to transmit information. The wave of large frequency (ν) is called the *carrier,* and the wave of small frequency ($\delta\nu$) is called the *envelope.* The envelope represents how the carrier is *modulated,*that is, how the amplitude is modified. This modulation contains the information that is transmitted.

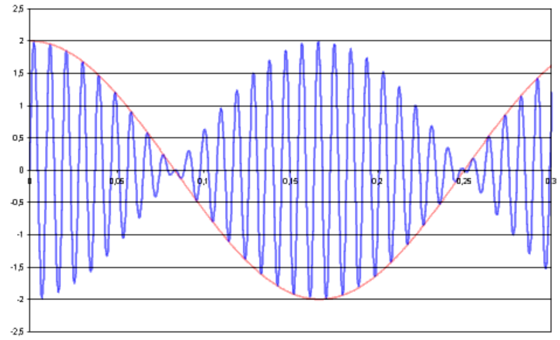

Figura 2: The blue curve represents the carrier wave. The red curve is the envelope and represents the beats.

As (1.6) is equivalent to (1.8), we can also derive the former from the latter. So the multiplicative superposition of two waves of respective frequency ν_1 and ν_2 is equivalent to the sum of two waves which have frequencies $\mu_1 = \frac{\nu_1+\nu_2}{2}$ and $\mu_2 = \frac{\nu_1-\nu_2}{2}$:

$$u(t) = A\sin(\nu_1 t + \varphi)\cos(\nu_2 t + \delta\varphi) = A[\sin(\mu_1 t + \varphi_1) + \sin(\mu_2 t + \varphi_2)]. \qquad (1.9)$$

Consonance and dissonance. This distinction mainly concerns simultaneous sounds, and indeed these notions arose especially in polyphonic Western music.

Consonance (or concord) and its opposite, namely dissonance, are subjective qualities of sound combinations. This is a rather complex notion, which involves acoustic, perception and cognitive science. We then distinguish *acoustic* consonance from *musical* consonance, although the boundary between these concepts is not sharp.

Beats are at the basis of acoustic dissonance. If two notes with sufficiently close frequencies ν_1 and ν_2 are played simultaneously, the onset of beating is rather disturbing if $\nu_1 - \nu_2$ falls in a critical range. But if ν_1 and ν_2 are either sufficiently close or sufficiently apart, then no beats are perceived, see Figure 3. This effect is used to tune musical instruments, since beats allow us to detect small variations in frequency that otherwise could not be perceived by our ear.

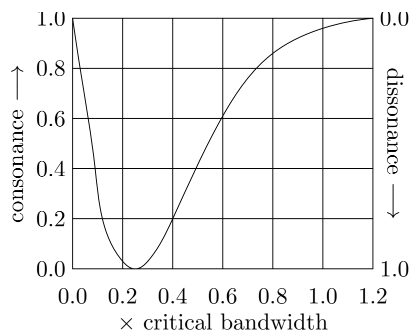

Figura 3: Acoustic consonance vs. frequency difference.

Acoustic dissonance may be regarded as objective. On the other hand, *musical dissonance* is subjective, conventional, and has a cultural basis. In also depends on the style and on the period. For instance, in the Renaissance just unison, octaves, perfect fifths and perfect fourths were regarded as musically consonant. [10] In the Common Practice Period major and minor thirds, major and minor sixths were also included in consonant interval. [11] The other intervals were considered dissonant to greater or lesser degree; for instance, an interval of a semitone is still regarded as highly dissonant.

Superficially, consonance might be regarded as pleasant and dissonance as disturbing. But prolonged consonance may be boring, and small amounts of dissonance can give flavor to musical compositions. A more advanced point of interpretation of this concept is that a consonant interval or chord actually seems stable and complete in itself, whereas a dissonance is perceived as incomplete, and wants to resolve to a consonance. We might loosely refer to consonance as rest, and dissonance as movement.

1.2 Some mathe-musical concepts [12]

Besides periodicity and the consequent decomposition in Fourier series, at the basis of music theory there is so-called *octave reduction.* This rests upon the logarithmic nature of our response to sound sensation.

Logarithmic frequency. Another example of the Fechner law of psychoacoustics states that (1.4) [13]

$$\text{the acoustic response of our auditory system depends linearly on the logarithm of the frequency of the impacting sound wave.} \tag{1.10}$$

For instance, let us either sing or play four pure sounds $S_1, ..., S_4$ respectively of frequency $f_1, ..., f_4$. If

$$f_2/f_1 = f_4/f_3 \qquad \text{i.e.,} \qquad \log f_2 - \log f_1 = \log f_4 - \log f_3,$$

we tend to perceive the interval (S_1, S_2) as musically similar to (S_3, S_4). So we express the difference between S_1 and S_2 by the frequency ratio f_2/f_1. On the logarithmic scale the interval S_1S_2 thus has length $\log f_2 - \log f_1 = \log(f_2/f_1)$. As we are more used to operate with differences rather than ratios, we shall represent frequencies on a logarithmic scale. More precisely, we shall call *pitch* the logarithm of the frequency of a pure tone. [14]

The choice of the basis of the logarithm is immaterial, because a change of basis corresponds to an irrelevant factor. Indeed

$$\log_a b \cdot \log_b c = \log_a c \qquad \forall a, b, c > 0,$$

since $a^{\log_a b \cdot \log_b c} = [a^{\log_a b}]^{\log_b c} = b^{\log_b c} = c = a^{\log_a c}$.

Octave reduction. Two pure tones sound at best together if the ratio of their frequencies is an integer power of 2. In music one calls *octave* (which is Latin for *eighth)* any frequency interval

[10] in 1324, Pope John XXII established by an edict that church music was allowed to use just these intervals.

[11] Here is a very rudimentary periodization of Western music, up to the end of the 19th century:
Early period, characterized by *modality:* Medieval era: 800–1400, Renaissance era: 1400–1600.
Common practice period, characterized by *tonality:* Baroque era: 1600–1750, Classical era: 1750–1820, Romantic era: 1820–1910. (Of course, this exercise is extremely schematic and disputable.)

[12] Music theory is permeated by (mostly simple) mathematical concepts: arithmetic, logarithms, equivalence relations, and so on. The last years have seen the birth and advancement of a mathematics of music *(mathe-music)* as a discipline characterized by peculiar problems and methods. However, music has not (yet) been given any axiomatic mathematical foundation.

[13] This can be explained by a model of our inner ear, in particular by the structure of the ear's *basilar membrane,* which acts as a sort of spectral analyzer. This fact is at the basis of our capability of recognizing a large range of sounds; e.g., at best we can hear frequencies between 20 and 20.000 Hz (about): a span of three orders of magnitude. (For an aged person the upper bound may be much smaller.)

[14] According to the Oxford Dictionary, *pitch* is the degree of highness or lowness of a tone. Although pitch depends on frequency, it is related to our acoustic perception, rather than to physical measurements. Pitch may be regarded as a sort of *psychoacoustic* counterpart of frequency. Here however by the term pitch we shall mean the logarithm of the frequency.

of length $L = \log 2$. Any frequency span can thus be decomposed into a (possibly non integer) number of octaves. [15]

For many purposes, in music

$$\text{one identifies two pure tones if the ratio of their frequencies is a power of 2.} \tag{1.11}$$

It is thus natural to select the basis 2 for the logarithm of the frequency. This identification is called *octave reduction.* The properties of equivalence relations are fulfilled, and one speaks of *octave equivalence.* [16] We shall call *pitch classes* these equivalence classes modulus 1.

The mathematical operation of octave reduction has a musical meaning: if one *transposes* (i.e., shifts the notes) of a musical piece, the music is perceived as similar. This is especially evident if the transposition is by an octave, i.e., if all frequencies are doubled. Something analogous occurs when a piece of music is played e.g. by a violin and a cello, or when a man and a woman sing the same song, or pronounce the same words, at an octave apart. (Voices however also differ in timbre, as we shall see.)

Moreover, although several musical elements depend just on pitch classes, other depend also on the octave. Logarithmic rescaling has a perceptual origin, whereas octave reduction seems to have a cultural basis. So, although octave reduction is shared by almost all cultures, it seems less mandatory than logarithmic rescaling. [17]

Note that several elements support the use of logarithms: because of the Fechner law (1.4) they have a perceptual basis, they allow to represent intervals by intervals instead of ratios, on the logarithmic scale octaves are equi-spaced.

Tones and keys. We intend to establish a correspondence between tones, frequencies and keys of the keyboard.

Figura 4: A keyboard.

First we define musical notes: in principle these are pure tones. [18] Typically the sound that is produced by musical instruments is a superposition of pure tones. If the sound is periodic, the pitch of the of the sound is typically associated to the fundamental frequency. Many instruments (e.g., keyboards, strings, woods, brasses) produce periodic sounds in which a component (typically, the fundamental) prevails over the other tones. For other instruments (typically percussion instruments like drums, bars, bells) inharmonic components prevail over the harmonics, and the sounds that they produce may look of indeterminate pitch.

[15] As we pointed out, human hearing sensibility ranges at best between 20 and 20.000 Hz (about). As $2^{10} = 1024 \simeq 1000$, this makes a maximum span of about 10 octaves. Human singing from bass to soprano has a total extension between 100 and 1000 Hz (about), so about four octaves.

The frequency range of a piano is between 27.5 Hz (A_0) and 4186 Hz (C_8), which is a span of more than seven octaves. The piano is one of the musical instruments with the largest extension; that of the organ is 9 octaves. (Organs include several keyboards: typically up to three or four *manuals* and a *pedal,* respectively operated by hands and feet. Anyway in New Jersey there is an organ with ten keyboards, for octopus players...)

[16] This is just one of the numerous algebraic structures that underly musical theory. Other examples of equivalence relation and corresponding identifications concern scale transposition, flats and sharps (so-called *enharmonicity),* modes, chord inversion, and so on. (Ahead we shall see some of these relations.)

[17] Anyway different cultures divide the octave differently. For instance, Indians divide it into 22 parts, Arabs into 17 parts, Chinese into 5 parts. We shall see that in the West we divide the octave into 12 semitones.

[18] The term *note* may also have other meanings in music: it may refer to a symbol on a musical score, to the tone as played by a musical instrument, to a key of a keyboard, and so on.

By looking at a keyboards, one immediately notices a repeated pattern of five black keys and seven white keys; this set is called an *octave,* see Figure 5. In a piano there are seven octave plus four extra keys, so 88 keys all together. [19]

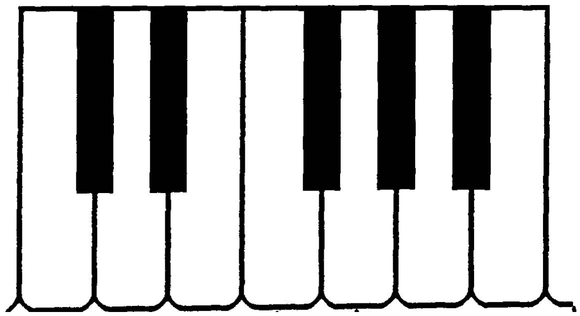

Figura 5: An octave on a keyboard.

The peculiar configuration of the octave needs and explanation. We shall provide a harmonic foundation for this black and white pattern by relating it to the Fourier decomposition of periodic sounds. Ahead, we shall also derive the exact values of the frequencies of the notes; this is the problem of *tuning.* This will need mathematical arguments, as well as musical criteria of selection based on consonance. As we shall see, several solutions have been proposed for this problem in the centuries, and the final answer is still somehow controversial. However, for the moment we intend to be more descriptive than deductive, and simply assume a correspondence between pure tones and keys, postponing any justification.

By octave reduction, we confine our attention to a single octave. Starting from the left side of the keyboard, we label the seven white keys as C, D, E, F, G, A, B, and the five black keys as $C^\sharp, D^\sharp, F^\sharp, G^\sharp, A^\sharp$, so that ordering by increasing frequency we have

$$C, C^\sharp, D, D^\sharp, E, F, F^\sharp, G, G^\sharp, A, A^\sharp, B. \tag{1.12}$$

(We shall see that we might also use flats instead of sharps.)

The chromatic clock. The 7 note on the white keys form a *diatonic* scale, the 5 note on the black keys form a *pentatonic* scale, and the 12 notes altogether form a *chromatic scale.* [20] The latter scale can be visualized on the *chromatic clock* (or *Krenek diagram),* see Figure 6. Other eleven chromatic scales are obtained by permuting cyclically the scale (2.18).

Fixing the exact frequencies of the notes is the purpose of the theory of *tuning* (or *intonation),* that we shall outline ahead. In order to simplify our task, here we assume that (as it is somehow suggested by the image of the clock) on the logarithmic scale

$$\text{the pitch difference of adjacent notes is approximately uniform.} \tag{1.13}$$

This difference is thus approximately 1/12; we call the corresponding frequency ratio $2^{1/12} \simeq 1.059463...$ a *semitone.* We then call *whole tone* (or just *tone)* the double of the semitone (on the logarithmic scale), which corresponds to the frequency ratio $2^{1/12} \times 2^{1/12} = 2^{1/6}$. We shall refer to (1.13) as the assumption of *approximate uniformity of the semitone.* As we shall see, this is exactly true just according to the theory of so-called *equal temperament.* However, a large part of music theory can be developed under the weaker assumption (1.13).

[19] The term *key* may also have other meanings in music: it may refer to the first note of a scale, to the key of a piece of music (which refers to the key of the scale in use), there are the violin and bass keys of the musical score, and so on. These terms will be defined ahead.

[20] In music there exists also several other scales, that here we shall not introduce.

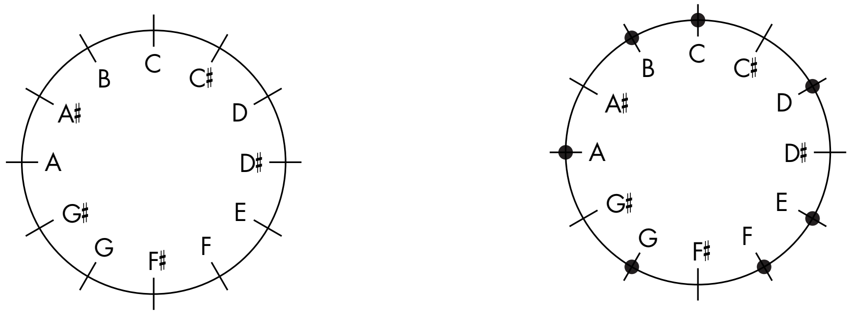

Figura 6: (a) 12-note *chromatic clock* (or Krenek diagram).
(b) The 7 diatonic notes are marked by bullets on the chromatic clock.

Hours are divided into minutes, whereas semitones are divided into *cents.* The octave consists exactly of 1200 cents, so that a semitone is approximately 100 cents, and the frequency ratio of the cent is $2^{1/1200} = 1.0005778....$ Therefore, within a single octave, denoting by δ_{CX} the distance between the pitch of a tone X and that of C measured in cents, we have

$$\delta_{CX} = [\log_2 \nu(X) - \log_2 \nu(C)] \times 1200 \text{ cents}, \qquad \text{i.e.} \qquad \nu(X)/\nu(C) \simeq 2^{\delta_{CX}/1200}. \tag{1.14}$$

Still within a single octave, denoting frequencies by ν, thus

$$\log_2 \nu(C^\sharp) - \log_2 \nu(C) \simeq ... \simeq \log_2 \nu(C) - \log_2 \nu(B) \simeq \frac{1}{12} = \frac{100}{1200} = 100 \text{ cents}. \tag{1.15}$$

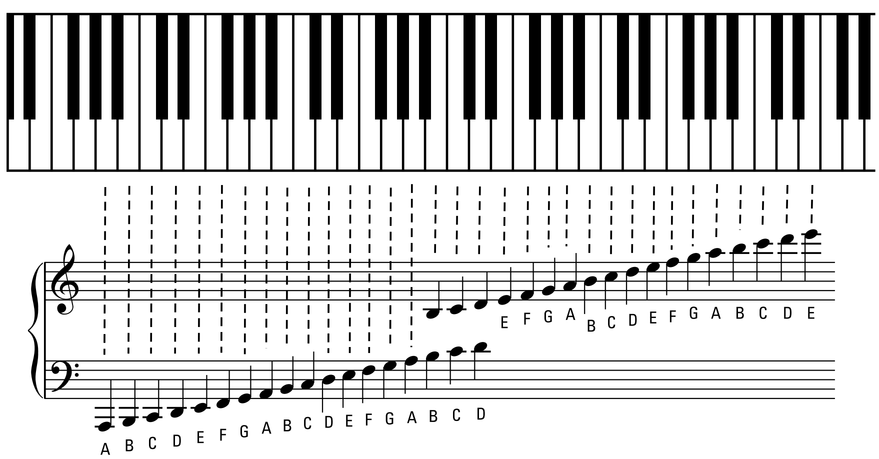

Figura 7: Correspondence between music notation on the *staves* and the notes on the standard keyboard, from A_1 to E_6.

On music notation. It has been noticed that, by introducing the staff about one thousand years ago, the monk Guido d'Arezzo [21] anticipated the Cartesian coordinates, and also introduced a first rudimentary example of *time-frequency analysis.* On the staves abscissas and coordinates respectively represent time logarithm of the frequency....

[21] This was somehow a legendary character: part of the many musical novelties that have been ascribed to Guido were already in use before his time.

1.3 Scales and intervals

Major and minor scales. 7-note diatonic scales play a central role in Western music. A large part of what we call *classical music* is essentially diatonic, that is, it mainly consists of the notes of a single diatonic scale. Many pieces however include some nondiatonic tones, to give flavor to the composition. In particular, this holds for the works of J.S. Bach, Haydn, Mozart, Beethoven (the main authors of the Baroque and Classical Periods), and also for most of the music of the Common Practice Period.

Scales are usually labelled by the first note, which is called the *key* of that scale. For any key there are two diatonic scales: a major and a minor scale. By octave reduction, the keys are 12, so there are 12 major diatonic scales and 12 minor diatonic scales, up to octave equivalence.

For instance, the (diatonic) C major scale consists of the pitch classes C, D, E, F, G, A, B, see Figure 8 (a). When one plays this sequence, usually one repeats the first note at the end, so that intervals between consecutive notes have the following lengths (here T = whole tone, S= semitone). By repeating the final note, one can also hear the interval between B and C. So a scale can be regarded as a finite sequence of notes, and *dually* as a sequence of intervals between adjacent notes. [22]

$$T, T, S, T, T, T, S. \tag{1.16}$$

This a sort of footprint of all major scales. [23]

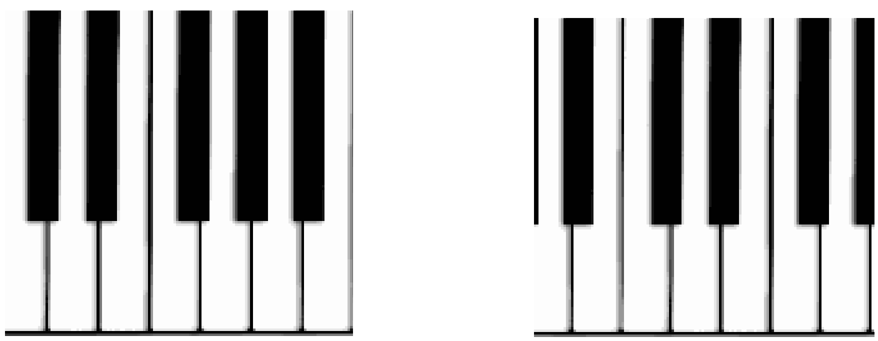

Figura 8: (a) Diatonic major scale of C (on the white keys).
(b) Diatonic minor scale of A (on the white keys).

The (diatonic) A minor scale consists of the pitch classes A, B, C, D, E, F, G, and is simply obtained by applying a ciclic permutation to the C major scale, see Figure 8 (b). So both scales correspond to the white keys of the keyboard: they consist of the same keys but have different musical flavour. Here the pattern of intervals between consecutive notes reads [24]

$$T, S, T, T, S, T, T. \tag{1.17}$$

The 48 major and minor scales are at the basis of the *tonal system.* There are also some variants of the minor scale, in particular the harmonic minor scales and the melodic minor scales, which here we shall not even define. There exist also other 7-note scales, in jazz for instance.

The major and minor scales in the other keys are obtained by *transposing* (i.e., translating notes by a fixed number of semitones) these two scales, preserving the respective sequences (1.16) and

[22] Maybe rather than to duality here one should refer to inversion: the note is a sort of discrete differential of the interval, and conversely the interval is a sort of sum of notes......

[23] According to the musical tradition of ancient Greece, this consists of two *major tetrachords, TTS*, separated by a tone,.

[24] This consists of two *minor tetrachord, TST*, followed by a tone.

(1.19) of the interval-lengths. For instance, the major and the minor scale of D respectively read

$$D, E, F^\sharp, G, A, B, C^\sharp, D, \qquad D, E, F, G, A, B, C^\flat, D.$$

(Sharps and flats are inserted, so that any diatonic note occurs in the scale.)

The C major scale and the A minor scale differ just by a permutation of the notes, and are called *relative scales.* On the other hand the C major scale and the C minor scale are called *parallel scales.* The same terminology is used for scales in the other keys: relative scales have the same notes, parallel scales have the same key. So in the major/minor system each scale has a relative scale and a parallel scale.

Modes. Any pair of relative scales are transposed of 3 semitones; e.g., A is distant 3 semitones lower than C. Two relative scales are also labelled as different *modes* of the same scale. [25] Actually, there are also other *modes,* for each key. For instance, there are seven modes of the scale of C, one for each of the seven cyclic permutations of C, D, E, F, G, A, B:

$$D, E, F, G, A, B, C, \quad E, F, G, A, B, C, D, \quad .... \quad B, C, D, E, F, G, A. \tag{1.18}$$

The second last is the minor scale of A. All correspond to the white keys of the keyboard. Obviously, these modes correspond to a cyclic permutations of the lengths of the intervals. The footprints of the modes (1.18) thus read

$$T, S, T, T, T, S, T, \quad S, T, T, S, T, T, T, \quad .... \quad S, T, T, S, T, T, T. \tag{1.19}$$

For each key there exists seven modes, which are obtained by transposing those that we just mentioned.

Quotient sets. In music theory one can find several equivalence relations with the corresponding quotient sets. Here are some examples.

(i) As we know the 12-note chromatic scale is obtained by octave reduction, it is thus the quotient with respect to an equivalence relation. [26]

(ii) Within a 12-note chromatic scale (so, after octave reduction) there are twelve 7-note diatonic scales, one for each of the 12 chromatic keys. [27] For each diatonic scale, seven *relative* modes are obtained by cyclic permutation. As cyclic permutations define an equivalence relation, each diatonic scale is a quotient set.

(iii) Two modes of different scales can be considered equivalent if they have the same *key.* (As we saw, one then says that the scales are mutually *parallel).* This defines an equivalence relation, with quotient set in bijection with the 12-note chromatic scale.

(iv) Two harmonic series (defined ahead) with different initial tones can be identified by relating harmonics of the same order. This is an equivalence relation. Here the quotient set is reduced to a singleton, so that one can speak of *the* harmonic series.

(v) The pattern (1.16) allows one to identify all major scales. Here also the quotient set is reduced to a singleton, so that one can speak of *the* major scale.

(vi) We shall see that, assuming equal temperament, notes with sharps and flats are identified: e.g., $C^\sharp = D^\flat$ (two notes like these are called *enharmonic).* Moreover, e.g., $C^\flat = B$, $C^{\sharp\sharp} = D$, $D^{\flat\flat} = C$, and other similar examples.

[25] Loosely speaking, *modality,* i.e. the use of the seven modes in C characterized the music of the Middle Age, the Renaissance was a transition period and in the Common Practice Period *tonality,* with the major/minor scales, replaced modality. Although the 20th century saw many new trends, including a revival of modality, the tonal system still pervades popular and commercial music.

Note that *major scale* and *minor scale* are misnomers: they actually are two different modes of the diatonic scale, as they consist of the same notes.

[26] This presumes that a tuning system has been selected.

[27] In passing note there exist also other (nondiatonic) 7-note scales, that we shall not consider in this survey.

Minor scales can also be identified, via the pattern (1.19). The same actually applies to each of the seven modes.

In music theory there are several other quotient sets. For instance, two intervals of the same length are often identified.

Intervals. Intervals play a major role in music. Although the ability to identify tones is rare, most people can identify intervals.

Intervals are of two types: melodic and harmonic. Melodic intervals corresponds to nonsimultaneus tones, whereas harmonic intervals corresponds to simultaneus tones. They are respectively represented horizontally and vertically on the musical score.

The denomination of the most relevant intervals of the chromatic scale is indicated below, jointly with their lengths expressed in semitones (s.t.):

0 s.t.	unison (e.g., C_4–C_4)	12 s.t.	octave (e.g., C_4–C_5)	
7 s.t.	perfect fifth (e.g., C_4–G_4)	5 s.t.	perfect fourth (e.g., C_4–F_4)	
4 s.t.	major third (e.g., C_4–E_4)	8 s.t.	minor sixth (e.g., C_4–$A_4^\flat$, A_4–F_4)	
9 s.t.	major sixth (e.g., C_4–A_4)	3 s.t.	minor third (e.g., C_4–$E_4^\flat$, A_4–C_4)	(1.20)
2 s.t.	major second (e.g., C_4–D_4)	10 s.t.	minor seventh (e.g., C_4–$B_4^\flat$, A_4–G_4)	
11 s.t.	major seventh (e.g., C_4–B_4)	1 s.t.	minor second (e.g., C_4–$D_4^\flat$).	

There is also the interval of 6 semitones (e.g., B_3–F_4). This is called tritone and is regarded as highly dissonant.

Notice that two intervals on the same line mutually complement to the octave.

Note that:

(i) the major second, third, sixth and seventh respectively correspond to the interval between the key and the second, third, sixth and seventh pitch of the major (diatonic) scale.

(ii) the minor third, sixth and seventh respectively correspond to the interval between the key and the third, sixth and seventh pitch of the minor (diatonic) scale. Note that this does not apply to the minor second. Any minor interval is a semitone shorter than the corresponding major interval (this also apply to the seconds).

(iii) the intervals between the key and the fourth and fifth pitch have the same length for the major and minor scales. For this reason they are called *perfect* intervals.

With reference to the classical style, the following intervals are regarded as consonant (ordered by degreasing consonance): unison, octave, 5th, 4th, 3rd, 6th. On the contrary, 2nd, tritone and 7th are considered dissonant. (Here we refer to intervals between notes played simultaneously.)

2 Tuning

The notes are the elements of the musical alphabet. In music notes play a role comparable to that of real numbers in mathematical analysis. Their construction, that is the definition of their frequency, is at the basis of the foundation of music.

2.1 The harmonic series

Two numerical transforms. Let us fix any frequency f_1, and transform nf_1 to the pitch

$$\log_2(nf_1) = \log_2 n + \log_2 f_1 \qquad \forall n \in \mathbf{N}.$$

Let us neglect the additive constant $\log_2 f_1$, and denote by $m = m(n) \in \mathbf{N}$ and $r = r(n) \in [0, 1[$ respectively the integer and fractional part of $\log_2 n$, so that

$$\log_2 n = m + r \quad \text{ with } \quad m \in \mathbf{N},\ 0 \le r < 1. \tag{2.1}$$

We thus have two functions:

$$\theta : \mathbf{N}f_1 \to [0,1] : nf_1 \mapsto r(n), \qquad \gamma : \mathbf{N}f_1 \to \mathbf{N} : nf_1 \mapsto m(n). \tag{2.2}$$

The function θ represents the distance between the tone of frequency f and the fundamental frequency f_1, after octave reduction. This consists in the following operations:
(i) normalize the frequency by dividing it by f_1, then
(ii) apply the logarithm in basis 2, finally
(iii) reduce the result modulus 1.
The *octave counting* function γ keeps trace of the specific octave the note belongs to.

The harmonic series. In music a *harmonic series* is the sequence of the harmonics of the first tone. [28] Thus, if f_n is the frequency of the nth tone, then $f_n = nf_1$.

The harmonic series can start from any frequency. All harmonic series are mutually equivalent by *transposition,* which corresponds adding a fixed real number to the logarithms of the frequencies, or equivalently multiplying all frequencies by a fixed positive number. This is obviously an equivalence relation. So by taking the quotient one can speak of *the* harmonic series.

We shall see whether one can construct the diatonic, pentatonic and chromatic scales via the harmonic series. (We anticipate that this will encounter some difficulties.) The function θ of (2.1) operates as follows:

$$\begin{aligned}
f_1 &\mapsto \log_2 1 = 0, \\
2f_1 &\mapsto \log_2 2 = 1 \overset{(\text{mod } 1)}{=} 0, \\
3f_1 &\mapsto \log_2 3 = 1 + \log_2(3/2) \overset{(\text{mod } 1)}{=} \log_2(3/2), \\
4f_1 &\mapsto \log_2 4 = 2 \overset{(\text{mod } 1)}{=} 0, \\
5f_1 &\mapsto \log_2 5 = 2 + \log_2(5/4) \overset{(\text{mod } 1)}{=} \log_2(5/4), \\
6f_1 &\mapsto \log_2 6 = 2 + \log_2(3/2) \overset{(\text{mod } 1)}{=} \log_2(3/2), \\
7f_1 &\mapsto \log_2 7 = 2 + \log_2(7/4) \overset{(\text{mod } 1)}{=} \log_2(7/4), \\
8f_1 &\mapsto \log_2 8 = 3 \overset{(\text{mod } 1)}{=} 0, \quad \text{and so on.}
\end{aligned} \tag{2.3}$$

By approximating the logarithms and evaluating the function γ, we get

$$\begin{aligned}
&\theta(f_1) \ = 0, && \gamma(f_1) \ = 0, \\
&\theta(2f_1) = 0, && \gamma(2f_1) = 1, \\
&\theta(3f_1) = \log_2(3/2) \simeq 7/12, && \gamma(3f_1) = 1, \\
&\theta(4f_1) = 0, && \gamma(4f_1) = 2, \\
&\theta(5f_1) = \log_2(5/4) \simeq 4/12, && \gamma(5f_1) = 2, \\
&\theta(6f_1) = \log_2(3/2) \simeq 7/12, && \gamma(6f_1) = 2, \\
&\theta(7f_1) = \log_2(7/4) \simeq 10/12, && \gamma(7f_1) = 2, \\
&\theta(8f_1) = 0, && \gamma(8f_1) = 3, \ \text{and so on.}
\end{aligned} \tag{2.4}$$

Here we rounded logarithms to multiples of 1/12, since the chromatic scale consists of 12 notes. Defining

$$\gamma(n) = \ \text{the closest integer to } \beta(nf_1) \qquad \text{for } n = 0, ..., 12, \tag{2.5}$$

we can then assign a chromatic note to each entry of the harmonic series, after octave reduction.

Harmonic series and scales. For the moment let us apply octave reduction.

[28] Here *series* is what in mathematics is called *sequence*. The sequence of the periods of the harmonic series is $\{T, T/2, ..., T/n, ...\}$, which is proportional to what in mathematics is called the harmonic sequence. Moreover, in mathematics the harmonic series is the divergent series $\sum_n n^{-1}$.

By (2.6),

$$\theta(2f_1) = 0, \tag{2.6}$$

$$\theta(3f_1) = \log_2(3/2) \simeq 7/12. \tag{2.7}$$

On the chromatic clock if we start from C and make 7 steps we get G, which corresponds to 7 o'clock. This is the 8th of the 12 chromatic notes, and is the 5th of the 7 diatonic notes. [29] With reference to the latter scale, one accordingly says that G is the *fifth of* C.

$$\theta(5f_1) = \log_2(5/4) \simeq 4/12. \tag{2.8}$$

On the chromatic clock if we start from C and make 4 steps we get E, which corresponds to 4 o'clock. This is the 5th of the 12 chromatic notes, and is the 3rd of the 7 diatonic notes. One says that E is the *major third of* C.

$$\theta(7f_1) = \log_2(7/4) \simeq 10/12. \tag{2.9}$$

Thus θ maps the frequency $7f_1$ to $\log_2(7/4) \simeq 10/12$. On the chromatic clock if we start from C and make 10 steps we get $A^\sharp$, which corresponds to 10 o'clock. This is the 11th of the 12 chromatic notes, and is no diatonic note.

The approximation $\log_2(7/4) \simeq 10/12$ is rather crude, more that the other approximations of (2.3). The actual pitch is intermediate between 9/12 and 10/12, so between A and $A^\sharp$, closer to $A^\sharp$ than to A. The pitch A is the 10th of the 12 chromatic notes, and the 6th of the 7 diatonic notes. One says that A is the *sixth of* C. Thus $A^\sharp$ is the projection (more precisely, the element of minimal distance on the logarithmic scale) of the seventh harmonic onto the chromatic scale, and A is its projection onto the diatonic scale.

Not only the approximation of the 7th harmonic is poor (for the 11th is even poorer), it is also somehow dissonant with the previous harmonics. [30] This should not be regarded as a drawback of the harmonic series, but rather as a shortcoming of the diatonic scale. [31]

Two examples of harmonic series. Let us fix a frequency f_1, and denote by H_n the nth note of the corresponding harmonic series. Here we do not apply octave reduction, and append the octave number to the notes as an index. For instance, if the *fundamental note* (i.e. the first note) of the harmonic series is C_2, here are the first 16 harmonic tones with their *chromatic* approximations:

$$\begin{aligned} &H_1 = C_2, \quad H_2 = C_3, \quad H_3 \simeq G_3, \quad H_4 = C_4, \\ &H_5 \simeq E_4, \quad H_6 \simeq G_4, \quad H_7 \simeq A_4^\sharp, \quad H_8 \simeq C_5, \\ &H_9 \simeq D_5, \quad H_{10} \simeq E_5, \quad H_{11} \simeq F_5^\sharp, \quad H_{12} \simeq G_5, \\ &H_{13} \simeq G_5^\sharp, \quad H_{14} \simeq A_5^\sharp, \quad H_{15} \simeq B_5, \quad H_{16} = C_6, ... \end{aligned} \tag{2.10}$$

By applying octave reduction and preserving the order, we get the sequence

$$\begin{aligned} &\widetilde{H}_1 = C, \quad \widetilde{H}_2 = G, \quad \widetilde{H}_3 = E, \quad \widetilde{H}_4 = A^\sharp, \\ &\widetilde{H}_5 = D, \quad \widetilde{H}_6 = F^\sharp, \quad \widetilde{H}_7 = G^\sharp, \quad \widetilde{H}_8 = B, ... \end{aligned} \tag{2.11}$$

The mismatch with the diatonic scale is evident. However, we shall see that the very first notes of the harmonic series provide the basis for the construction of this scale.

[29] This corresponds to counting by ordinals instead of cardinals. For instance a fifth plus a third is a seventh, although by counting the semitones we have $4 + 2 = 6$. We use a similar terminology for centuries: for instance, the '800 is called the 19th century.

[30] Hindemith even refused to call *harmonic* this as well as higher components of the harmonic series.

[31] If we desired a fully harmonic scale, we might deal with the 8-note scale that is provided by the first harmonic up to $2^8 f_1$. Apart the historical reasons, why don't we do so? Maybe so harmonic a scale would be too boring?

The approximation of the pentatonic scale is better. After reordering, the first 5 pitch classes of (2.11) are $C, D, E, G, A^\sharp$. If we replace $A^\sharp$ by A (we saw that both approximate the real tone), we get C, D, E, G, A. This is called a (minor) pentatonic scale, and is the first scale that was discovered by the Chinese more than 2000 year ago. [32]

One can repeat the same procedure starting from any other note, or equivalently one can transpose the harmonic series above.

Let us consider another example. As $\nu(A_4)$ was fixed at 440 Hz, [33] computations are especially simple by selecting A_2 (i.e., the A of the second octave) as fundamental note. A calculation like that above shows that the harmonic series of A_2 reads

$$A_2, \quad A_3, \quad E_4, \quad A_4, \quad C_5^\sharp, \quad E_5, \quad G_5, \quad A_5, ... \tag{2.12}$$

The corresponding frequencies are exactly

$$\begin{aligned} &\nu(A_2) = 110\ Hz, \ \ \nu(A_3) = 220\ Hz, \ \ \nu(E_4) = 330\ Hz, \ \ \nu(A_4) = 440\ Hz, \\ &\nu(C_5^\sharp) = 550\ Hz, \ \ \nu(E_5) = 660\ Hz, \ \ \nu(G_5) = 770\ Hz, \ \ \nu(A_5) = 880\ Hz, ... \end{aligned} \tag{2.13}$$

These are in the ratios $\nu(A_2) : \nu(A_3) : ... : \nu(A_5) = 1 : 2 : ... : 8$, consistently with the definition of the harmonic series. [34]

Remarks. (i) In the construction of the harmonic series we introduced two approximations: we assumed that the length of the semitone is approximately uniform, see (1.13), and rounded the function θ, see (2.6). This is not surprising, since we were transforming the continuous family of frequencies to a finite set of notes.

(ii) For any integer n, the number of entries of the harmonic series up to $2^n f_1$ is 2^n. The sequence can also be divided into batches of consecutive octaves. The batch between $2^n f_1$ and $2^{n+1} f_1$ includes 2^n entries, which are representatives of all the pitch classes up to $2^{n+1} f_1$. Therefore the entries up to $2^{n+1} f_1$ comprise exactly 2^n pitches.

About musical instruments. As we saw, the pitch of a T-periodic sound is related to the fundamental frequency $f_1 = 1/T$. The timbre of a musical instrument or of a human voice depends on the ratios $\{I_k/I_1 : k \in \mathbf{N}\}$ between the intensity of the k-th harmonics and that of the fundamental harmonic. These ratios determine how the acoustic power is distributed among the harmonics. A tuning fork just produces a *monochromatic* signal, namely a sound of a single frequency, namely a pure tone without overtones. The sing and the sound of musical instruments are richer in harmonics. (The more they are, the more the sound looks bright. The less they are, the more the sound looks warm or dark.) On the other hand, our acoustic perception does not distinguish differences in phase of different harmonics. This depends on the human physiology of hearing, namely the inner ear and the brain.

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[32] By raising these frequencies of 6 semitones, this scale is transposed to $F^\sharp, G^\sharp, A^\sharp, C^\sharp, D^\sharp$, which correspond to the five black keys of a keyboard. The harmonic origin of the pentatonic scale explains why any tune played on the black keys is pleasant to the hear.

[33] This was an official decision taken at various international conferences in 1939, 1953, 1955. In the Baroque period the central A varied somehow between 400 and 450 Hz.

In Italy this is prescribed by a law of the state, which also establishes a fine for transgressors. The Legge 3 maggio 1989 reads: "Articolo 5 : L'utilizzazione di strumenti di riferimento non conformi all'articolo 3 [frequenza di 440 Hz del "la" centrale] è punita con una sanzione amministrativa per ogni esemplare da lire centomila a lire un milione." This is how out-of-tune is prosecuted in the *Bel Paese.*

Nowadays many symphonic orchestras tune to A = 443 Hz; many modern ensembles which specialize in the performance of Baroque music have agreed on a standard of A = 415 Hz, which is one semitone flatter than A = 440 Hz (absolutely illegal ...).

[34] The harmonic series that is generated by F reads $F, F, C, F, A, C, E, F, G, A, B^\flat, D, ...$ By excluding repetitions and by reordering, we get a seven-note scale: $C, D, E, F, G, A, B^\flat$; this is called the *harmonic scale* of C. (It should not be confused with the *harmonic minor scale* of C, see ahead)

For instance, a clarinet produces mainly odd-numbered harmonics, whereas a guitar generates even as well as odd harmonics. This is due to the structure of these instruments: a tube open at one end for the clarinet, strings tied at both ends for the guitar. The series (1.1) must converge; in many cases the intensity of the harmonic components decreases as the frequency increases, but it is not always so. For instance, for pianos the intensity rapidly decays from the first to the second component, and then decays more slowly. [35] As we pointed out, the sound of a real musical instrument may also include inharmonic components. For instance, for the sound produced by a piano includes the harmonic components generated by the strings and by the (wooden) harmonic table, as well as the inharmonic components due to the stiffness of the strings and of the wooden frame.

About keyboards. Let f_1 be the nominal frequency of a key. By this we mean that playing the key generates a wave of fundamental frequency f_1, which includes overtones of frequency $f_n = nf_1$ for $n = 1, 2, ...$ Moreover, the strings that have fundamental frequency nf_1 will also resonate, if they are not constrained. By pressing the *sustain* pedal (the one on the right), the pianist allows the other keys to resonate, by removing a constraint. This enhances the *color* of the performance.

For instance, on an acoustic piano first one can play the key G_3, the third harmonic of C_2. Next one can gently press the key C_2 without producing any sound; while keeping that key pressed, one can then play the key G_3 again. As one can perceive, the sound that is now produced by G_3 is richer than the previous one. This effect can be increased by keeping the *sustain pedal* of the piano pressed.

As we saw, if for instance f_1 corresponds to the note A_2, then the frequencies $2f_1, 4f_1$ respectively correspond to the notes A_3 and A_4, of the same pitch class. A different pitch class occurs just by the third harmonic, which is the note E_3 in this case. Other A_n's, E_n's and other pitch classes follow with higher harmonics. Therefore by pressing that key produces several pitches, but the intensity of the pitch class A clearly prevails, so that we have good reason for associating this pitch class to that key.

By (2.3), just higher octaves can resonate. This has the following acoustic asymmetric effect, which can be clearly perceived by the ear. By pressing a key of a lower octave (i.e., at the left side of the keyboard), one triggers the resonance of the higher octaves; this makes the sound warm. On the other hand, by pressing a key of a higher octave (i.e., at the right side of the keyboard), less strings can resonate, and the sound is less colorful.

2.2 Pythagorean tuning

The term *tuning* (or *intonation* or *temperament)* indicates a system that fixes the frequencies of the tones of a musical scale exactly, making more precise the assumption (1.13). Here we outline the Pythagorean and Ptolomaic tunings, and equal temperament.

Small ratios. If the ratio between the fundamental frequencies of two tones can be represented as a fraction of two relatively small integer numbers (e.g., 5/4 but not 234/128), [36] then these sounds share a certain number of overtones of relatively low order. This tends to make the overlapping of these sounds agreeable, consistently with the theory of consonance that we outlined above. (Notice that two overtones may have close frequencies, but this cannot occur for low order overtones.)

Overtones of low order are the most relevant, since most often they have larger amplitude than overtones of high order, and thus they acoustically prevail over the latter. The degree of this prevalence depends on the musical instrument under consideration, be it the human voice or a mechanic or electronic instrument. Typically, this is determined by the very first overtones. In

[35] By *piano* we shall refer to the traditional acoustic (or mechanical) piano. This may be an *upright* (or *vertical) piano,* or a *grand piano;* the latter is heavier and larger, and is used in concerts. Digital pianos simulate the sound of mechanical pianos, but they play sampled sounds, rather then producing them mechanically.

[36] Musicians call these ratios *small ratios.* So for instance 7/2 is a small ratio, but 2/1000 is not. They also call 7/2 an irrational number, because it is a fraction. (A world apart...)

particular, as we saw, three of the first four harmonics of a sound are in the same pitch class as the fundamental tone, and this suffices for ascribing that pitch class to that sound. [37]

For instance, two pitch classes that are a fifth apart, e.g. C and G, have frequency ratio $3/2$, since $2\nu(G_n) = 3\nu(C_n)$ for any $n \in \mathbf{N}$. Therefore

$$2m\nu(G_n) = 3m\nu(C_n) \qquad \forall m \in \mathbf{N}, \forall n \in \mathbf{Z}, \tag{2.14}$$

that is, the frequency of the overtone of G_n of order $2m$ coincides with the frequency of the overtone of C_n of order $3m$; see Figure 9. Actually, C_n and G_n are low order harmonics of C_{n-1}, since, by

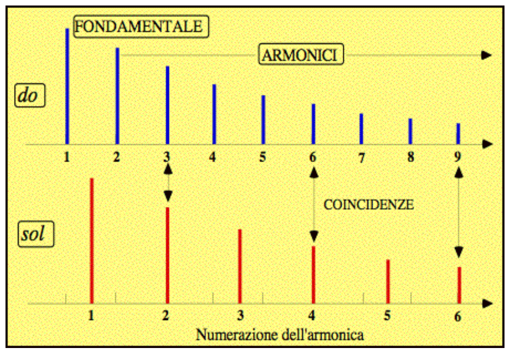

Figura 9: Harmonics of C and G, represented by the distribution of power as a function of the logarithm of the frequency.

octave equivalence

$$\nu(C_n) = 2\nu(C_{n-1}), \qquad \nu(G_n) = 2\nu(G_{n-1}) \overset{(2.14)}{=} 3\nu(C_{n-1}) \qquad \forall n \in \mathbf{Z}.$$

The Pythagorean tuning. Pythagoras (ca. 569 BC - ca. 475 BC), and the Chinese before him, constructed musical scales. Of course they ignored Fourier series, and considered length proportions of rudimentary musical instruments: strings for Pythagoras, bamboo pipes for Ling Lun, the legendary founder of music in ancient China. Pythagoras calculated the mathematical ratios of musical intervals using a *monochord,* namely, a string fixed at its ends. By fixing the string also at an intermediate point, and then plucking one of the two parts, he was able to produce sounds of different frequencies. E.g., by halving the length of the chord, the period is halved, namely the frequency is doubled; by reducing the length to $1/3$, the period is divided by 3, namely the frequency is multiplied by 3; and so on.

The Pythagorean scale only used the ratios $2/1$ (octave), $3/2$ (perfect fifth), and the complement of the latter to the octave $4/3$ (perfect fourth), since only these intervals were regarded as consonant in ancient music. So the notes that they considered have frequency ratios of the form $2^m 3^n$, with $m, n \in \mathbf{Z}$. In the Renaissance further intervals were included among the consonant ones, and this led to the formulation of *just tuning,* see ahead.

The progression of the fifths. Let us move from the pure tone F_3, without assuming octave reduction. As we saw in (2.6), along the sequence of the overtones of F_3, the first new pitch class is encountered with the third harmonic, C_4. As we saw, this is a fifth higher than F_3, and (denoting the frequency of F_3 by f_0) has pitch

$$\log_2(3f_0) = \log_2 3 + \log_2 f_0 = 1 + \log_2(3/2) + \log_2 f_0 \overset{(\text{mod } 1)}{=} \log_2(3/2) + \log_2 f_0.$$

[37] If the amplitude of the components decreases with the order, the fifth has just the bronze medal, whereas on the side of the fundamental there are golden and silver medalists, with the first of the nonmedalists: the prevalence is overwhelming.

Along the harmonic series the frequency varies through an arithmetic progression of multiples of f_0: $2f_0$, $3f_0$, ..., nf_0,.... Here we let the frequency vary through a geometric progression of common ratio 3/2, namely f_0, $(3/2)f_0$, ..., $(3/2)^n f_0$,.... In this way we fix the frequency of successive fifths exactly.

Neglecting the additive constant $\log_2 f_0$, the mapping from frequency to pitch class reads

$$\begin{aligned}
f_0 &\mapsto \quad 0, \\
(3/2)f_0 &\mapsto \quad \log_2(3/2), \\
(3/2)^2 f_0 &\mapsto \quad \log_2(9/4) = 1 + \log_2(9/8) \overset{(\text{mod } 1)}{=} \log_2(9/8), \\
(3/2)^3 f_0 &\mapsto \quad \log_2(27/8) = 1 + \log_2(27/16) \overset{(\text{mod } 1)}{=} \log_2(27/16), \\
(3/2)^4 v &\mapsto \quad \log_2(81/16) = 2 + \log_2(81/64) \overset{(\text{mod } 1)}{=} \log_2(81/64), \\
(3/2)^5 v &\mapsto \quad \log_2(243/32) = 2 + \log_2(243/128) \overset{(\text{mod } 1)}{=} \log_2(243/128), \\
(3/2)^6 f_0 &\mapsto \quad \log_2(729/64) = 3 + \log_2(729/512) \overset{(\text{mod } 1)}{=} \log_2(729/512),
\end{aligned} \tag{2.15}$$

and so on. This yields the following correspondence between frequencies and notes:

$$\begin{aligned}
f_0 &\mapsto \quad F_3, \\
(3/2)f_0 &\mapsto \quad C_4, \\
(3/2)^2 f_0 &\mapsto \quad G_4, \\
(3/2)^3 f_0 &\mapsto \quad D_5, \\
(3/2)^4 f_0 &\mapsto \quad A_5, \\
(3/2)^5 f_0 &\mapsto \quad E_6, \\
(3/2)^6 f_0 &\mapsto \quad B_6.
\end{aligned} \tag{2.16}$$

By further progressing by fifths, we determine the frequency of pitch classes with sharps: $F^\sharp, C^\sharp, G^\sharp$, and so on. After reordering, according to the Pythagorean tuning we get the following ratios between the frequencies of the diatonic notes of the octave and the frequency of the fundamental note:

$$\begin{aligned}
&\frac{\nu(D)}{\nu(C)} = \frac{9}{8}\,, \quad \frac{\nu(E)}{\nu(C)} = \frac{81}{64}\,, \quad \frac{\nu(F)}{\nu(C)} = \frac{4}{3}\,, \\
&\frac{\nu(G)}{\nu(C)} = \frac{3}{2}\,, \quad \frac{\nu(A)}{\nu(C)} = \frac{27}{16}\,, \quad \frac{\nu(B)}{\nu(C)} = \frac{243}{128}\,.
\end{aligned} \tag{2.17}$$

Here we find just two lengths for intervals between consecutive notes: $\log_2(9/8)$ (whole tone) and $\log_2(256/243)$ (semitone). But the whole tone is not the double of the semitone, as $\log_2(9/8) \neq 2\log_2(256/243)$.

On the other side, regressing by fifths from F_3, one gets $B_2^\flat, E_2^\flat, A_1^\flat, D_1^\flat, G_0^\flat$. This completes the construction of the 12-tone chromatic scale, see Figure 10:

$$..., G^\flat, D^\flat, A^\flat, E^\flat, B^\flat, F, C, G, D, A, E, B, F^\sharp, ... \tag{2.18}$$

In principle, the progression goes to infinity on either side. Every step of the progression produces a new note, which does not coincide with any of those that had already been defined. In other terms, the notes of the progression is not periodic. So the progression moves along a spiral. We shall get a circle just with equal temperament. (Just tuning is not comparable, since there we have two generators: fifths and (major) thirds; so we have two directions of movement.)

Alternative progressions. As $\log_2[(3/2)^{-1}] \overset{(\text{mod } 1)}{=} \log_2(4/3)$, by octave reduction the inverse of a fifth is a fourth:

$$\text{5th up} \;\Leftrightarrow\; \text{4th down} \qquad\qquad \text{4th up} \;\Leftrightarrow\; \text{5th down}. \tag{2.19}$$

Rather than stacking fifths, Pythagoras equivalently proceeded by alternating fifths forwards, $\overset{5th}{\nearrow}$, with fourths backward, $\overset{4th}{\searrow}$. (This also allowed Pythagoras to define the whole tone.) In this way, still starting e.g. from F_3, one gets

$$F_3 \overset{5th}{\nearrow} C_4 \overset{4th}{\searrow} G_3 \overset{5th}{\nearrow} D_4 \overset{4th}{\searrow} A_3 \overset{5th}{\nearrow} E_4 \overset{4th}{\searrow} B_4. \tag{2.20}$$

These notes are confined to one octave (from F_3 toE_4). One might equivalently proceed to the left of F_3 alternating fourths forward, $\overset{4th}{\nearrow}$, with fifths backward, $\overset{5th}{\searrow}$.

In principle, the progression goes to infinity on either side. Every step of the progression produces a new note, which does not coincide with any of those that had already been defined.

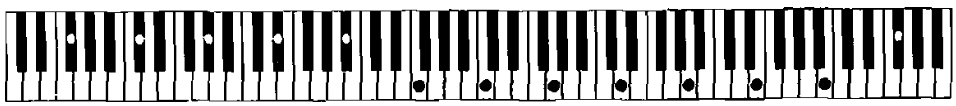

Figura 10: Generation of the chromatic scale by progression of the fifths.

Remarks. (i) Let us consider the extended version of the progression of the fifths:

$$..., F^\flat, C^\flat, G^\flat, D^\flat, A^\flat, E^\flat, B^\flat, \quad F, C, G, D, A, E, B, \quad F^\sharp, C^\sharp, G^\sharp, D^\sharp, A^\sharp, E^\sharp, B^\sharp, ... \tag{2.21}$$

Here we just displayed three batches of 7-note diatonic sequences. Further left we get the double flats: $F^{\flat\flat}, ..., B^{\flat\flat}$, whereas further right we have the double sharps: $F^{\sharp\sharp}, ..., B^{\sharp\sharp}$.

Note that the notes of any pair of these consecutive batches are shifted by a semitone.

(ii) This progression generates the chromatic, diatonic pentatonic scales in the sense that next we explain.

Starting from any tone, each batch of 12 consecutive tones forms a chromatic scale. All twelve chromatic scales are generated in this way.

Starting from any tone, each batch of 7 consecutive tones forms the elements of a diatonic scale; more precisely, the second element of the batch is the key of that scale. All seven diatonic scales are generated in this way. For instance, by reordering,

$$\begin{aligned} &F, C, G, D, A, E, B \quad \to \text{ diatonic scale of } C, \\ &C, G, D, A, E, B, F^\sharp \quad \to \text{ diatonic scale of } G, \\ &G^\flat, D^\flat, A^\flat, E^\flat, B^\flat, F, C \quad \to \text{ diatonic scale of } D^\flat. \end{aligned} \tag{2.22}$$

Similarly, each batch of 5 consecutive tones forms the elements of a pentatonic scale. All five pentatonic scales are generated in this way.

(The above properties of symmetry stem from the translational invariance of the progression of the 5ths, and in turn this is based on the analogous symmetry of the harmonic series.)

2.3 Ptolomaic tuning

Ptolomaic just tuning. Up to all the Middle Age octaves, perfect fifths and perfect fourths were regarded as the only consonant intervals, and the Pythagorean tuning was used. In the Renaissance thirds started to be considered consonant, and in the 16th century Gioseffo Zarlino proposed a tuning that Ptolomeus had formulated in the 2th century a.C.. [38]

[38] Claudius Ptolomeus was a mathematician and an astronomer. He is especially known for his geocentric model of the universe, which was the paradigm of astronomy untill Copernicus formulated his heliocentric theory. (As we know, in ancient times music mathematics and astronomy were strictly related. The great astronomer Johannes Kepler also strictly related astronomy and music.)

The Pythagorean tuning proceeds by (perfect) fifths, with no concern for (major) thirds, which are defined by the ratio 81/64 just as an indirect outcome of the procedure, see (2.17). On the other hand, the harmonic series yields the *small ratio* 5/4 (= 80/64) for thirds. The ratio between these two fifths is 81/80, which from the musical point of view is not negligeable. The Ptolomaic tuning, combines progression in two directions: by (perfect) fifths and (major) thirds. This corresponds to representing frequencies ratios in the form $2^m 3^n 5^p$, with $m, n, p \in \mathbf{Z}$. [39] The use of three prime numbers instead of two yields smaller ratios than those of the Pythagorean system.

Let us consider four consecutive fifths: F, C, G, D. After octave reduction, they correspond to the frequency ratios

$$\frac{\nu(F)}{\nu(C)} = \frac{4}{3}\,, \qquad \frac{\nu(G)}{\nu(C)} = \frac{3}{2}\,, \qquad \frac{\nu(D)}{\nu(C)} = \frac{\nu(D)}{\nu(G)} \cdot \frac{\nu(G)}{\nu(C)} = \frac{9}{4} \overset{(\text{mod } 1)}{=} \frac{9}{8}\,.$$

Next we progress by third from F, C and G, getting respectively A, E and B, which correspond to the frequency ratios

$$\frac{\nu(A)}{\nu(C)} = \frac{\nu(A)}{\nu(F)} \cdot \frac{\nu(F)}{\nu(C)} = \frac{5}{4} \cdot \frac{4}{3} = \frac{5}{3}\,, \qquad \frac{\nu(E)}{\nu(C)} = \frac{5}{4}\,,$$

$$\frac{\nu(B)}{\nu(C)} = \frac{\nu(B)}{\nu(G)} \cdot \frac{\nu(G)}{\nu(C)} = \frac{5}{4} \cdot \frac{3}{2} = \frac{15}{8}\,.$$

After reordering we get the following ratios

$$\begin{aligned} &\frac{\nu(D)}{\nu(C)} = \frac{9}{8}\,, \quad \frac{\nu(E)}{\nu(C)} = \frac{5}{4}\,, \quad \frac{\nu(F)}{\nu(C)} = \frac{4}{3}\,, \\ &\frac{\nu(G)}{\nu(C)} = \frac{3}{2}\,, \quad \frac{\nu(A)}{\nu(C)} = \frac{5}{3}\,, \quad \frac{\nu(B)}{\nu(C)} = \frac{15}{8}\,. \end{aligned} \tag{2.23}$$

This construction produces the same ratios starting from any other diatonic note. We can thus compare the two sequences (2.17) and (2.23). In the latter the third and the sixth are defined by smaller ratios, respectively 5/4 and 5/3, thus they are more harmonic than in (2.17). In (2.23) the minor thirds EG and AC also have a small ratio, 6/5. However, the minor third DF has the less consonant ratio: 32/27.

Here we have a semitone of ratio 16/15, and two whole tones of ratios 9/8 and 10/9, and neither of the latter is the double of the semitone.

Tuning the minor scale. Diatonic *triads,* namely chords of three diatonic notes, are at the basis of the Western harmony, since the 18th century. The notes of the *major triads,* FAC, CEG and GBD are in the ratio $1 : 5/4 : 3/2$, whereas the notes of the *minor triads* are in the following ratios

$$FAC \to 4/3 : 5/3 : 2, \quad CEG \to 1 : 5/4 : 3/2, \quad ACE \to 3 : 15/4 : 9 : 4.$$

In (2.23) we tuned the octave using fifths and *major* thirds, and thus we derived the frequency ratios for the major diatonic scale of C. As an alternative, one can tune the octave using fifths and *minor* thirds, and thus derive the frequency ratios for the minor diatonic scale of A. This yields the following (small) ratios

$$\begin{aligned} &\frac{\nu(B)}{\nu(A)} = \frac{9}{8}\,, \quad \frac{\nu(C)}{\nu(A)} = \frac{6}{5}\,, \quad \frac{\nu(D)}{\nu(A)} = \frac{4}{3}\,, \\ &\frac{\nu(E)}{\nu(A)} = \frac{3}{2}\,, \quad \frac{\nu(F)}{\nu(A)} = \frac{8}{5}\,, \quad \frac{\nu(G)}{\nu(A)} = \frac{9}{5}\,. \end{aligned} \tag{2.24}$$

[39] We can thus associate this tuning with the multiplicative subgroup of $\mathbf{R}^+$ generated by $2, 3, 5$, and that generated by $2, 3$ for the Pythagorean tuning.

These frequencies of the minor scale are consistent with those of the major scale, see (2.23). For instance, $\frac{\nu(C)}{\nu(A)}\frac{\nu(A)}{\nu(C)} = \frac{6}{5}\frac{10}{6} = 2$. Merging the frequencies of the major and minor scales, one can easily get those of the chromatic scale. We leave this task to the reader. (For instance, $\frac{\nu(E^\flat)}{\nu(C)} = \frac{\nu(C)}{\nu(A)} = \frac{6}{5}$.)

2.4 Equal temperament

Drawbacks of Pythagorean and Ptolomaic tunings. By the Pythagorean tuning, having produced more than 12 pitch classes, one might wonder whether two of them coincide. As pitches were constructed iteratively, this would entail that the progression is periodic. But

$$(3/2)^{12} \simeq 129.75 > 128 = 2^7\,,$$

12 (perfect) Pythagorean fifths are slightly sharper than 7 octaves. The frequency ratio between 12 Pythagorean fifths and 7 octaves is called the *Pythagorean comma,* and equals

$$\nu(F^\sharp)/\nu(G^\flat) = (3/2)^{12}/2^7 = 3^{12}/2^{19} = 1.01364... \tag{2.25}$$

By (1.14), on the logarithmic scale this corresponds to the distance

$$[\log_2 \nu(F^\sharp) - \log_2 \nu(G^\flat)] \times 1200 = [\log_2(1.01364...)] \times 1200 = 23.5... \text{ cents.} \tag{2.26}$$

i.e., almost a quarter of semitone.

The progression of the fifths is not closed, that is, it is not periodic. If it were so, then after a suitable number of fifths one would retrieve the same note with a frequency ratio equal to an integer power of two. This is impossible, since

$$(3/2)^h \neq 2^k \qquad \forall h, k \in \mathbf{N}, \tag{2.27}$$

namely, $3^h \neq 2^{h+k}$ for any integer h, k. The same conclusion holds for the Ptolemaic just tuning, since

$$\forall m, n, p, h \in \mathbf{Z}, \quad np \neq 0 \quad \Rightarrow \quad 2^m 3^n 5^p \neq 2^k. \tag{2.28}$$

The same applies to the progression of the thirds. For instance, starting by C_4 and progressing by thirds, one would get the ordering $C_4, E_4, G_4, B_4, D_5, F_5, A_6$. This yields qualitatively similar outcomes, with the same diatonic scale, but each with slightly different frequencies, and still with unequally spaced pitches. In conclusion, none of these methods nor the many variants that were suggested over the centuries can overcome the drawbacks we just illustrated.

Equal temperament. A way of overcoming these difficulties was provided by equal temperament, as the outcome of a long historical process. Equal temperament had been known in China for a long time, and was officially adopted in 1596. It was first proposed in the West by Aristoxenos of Taranto about 320 b.C., it was also advocated e.g. by Vincenzo Galilei (father of Galileo) in 1581, and is sometimes improperly ascribed to a work of Werckmeister of 1691. It gradually entered Western music practice at the end of Renaissance.

By equal temperament the octave is divided into a family of equally spaced intervals (on the logarithmic scale, of course). In this way the Pythagoren comma is uniformly spread over the octave, or equivalently over the interval $G^\flat$-$F^\sharp$, so that $G^\flat = F^\sharp$ up to octave reduction. This would not be possible by applying rational increments of frequency. As we saw, the ratio between the frequencies of two contiguous Cs is 2, and there are 12 semitones in each octave; the uniform ratio between the frequencies of two tones that are a semitone apart should then be $2^{1/12}$, which is irrational.

As a fifth, a major third and a minor third respectively correspond to 7, 4 and 3 semitones, with equal temperament these ratios are replaced by

$$7 \times 2^{1/12}, 4 \times 2^{1/12}, 3 \times 2^{1/12}, \text{ respectively,} \tag{2.29}$$

preserving their mutual proportion, as it occurs in just tuning.

It is not easy to find a mechanical device for tuning with equal temperament. Several instruments are tuned *by ear,* striking a compromise between the harmonicity of fifths and thirds.

The circle of the fifths. Equal tuning has several advantages: in this set-up no approximation is needed to close the progression of the fifths. Starting from $G^\flat$ one gets $G^\flat, D^\flat, ..., B, F^\sharp, ...$ as in (2.21). But, at variance with the Pythagoren tuning, here $F^\sharp = G^\flat$, closing the circle. The 12 note pattern

$$F^\sharp = G^\flat, D^\flat, A^\flat, E^\flat, B^\flat, F, C, G, D, A, E, B \tag{2.30}$$

is thus indefinitely repeated on both sides. This is called the *circle of the fifths.*

Note that here "the sharps coincide with the flats", in the sense that

$$C^\sharp = D^\flat, D^\sharp = E^\flat, F^\sharp = G^\flat, G^\sharp = A^\flat, A^\sharp = B^\flat. \tag{2.31}$$

Moreover, double flats and double sharps can be dropped:

$$C^{\sharp\sharp} = C, D^{\sharp\sharp} = D, ... \quad C^{\flat\flat} = C, D^{\flat\flat} = D, ... \tag{2.32}$$

By using just twelve notes, one can thus transpose musical pieces without retuning keyboards. There is less variety, but more order, and of course this is much more practical. [40]

Anyway equal temperament has a price: one reproduces the harmonic scale only approximately. We associated this scale to the Fourier expansion with frequencies that are in integer ratios: here all that is lost, or at least is just an approximation. From this point of view, every tone is out of tune. But the error is spread uniformly over the twelve pitches. (Some musicians object that in this way we can play everything, but we play it poorly.)

Table 25.1 Twelve-Toned Chromatic Scale Tuned in Equal Temperament, Just Intonation, and Pythagorean Intonation

Equally Tempered		Just Intonation		Pythagorean Intonation	
Note	**Cents**	**Ratio**	**Cents**	**Ratio**	**Cents**
C	0000	1/1	0000	1/1	0000
Db	0100	16/15	0112	256/243	0090
D	0200	9/8	0204	9/8	0204
Eb	0300	6/5	0316	32/27	0294
E	0400	5/4	0386	81/64	0408
F	0500	4/3	0498	4/3	0498
F#	0600	45/32	0590	729/512	0612
G	0700	3/2	0702	3/2	0702
Ab	0800	8/5	0814	128/81	0792
A	0900	5/3	0884	27/16	0906
Bb	1000	9/5	1018	16/9	0996
B	1100	15/8	1088	243/128	1110
C	1200	2/1	1200	2/1	1200

Figura 11: Comparison of temperaments.

Mini English-Italian musical vocabulary:

[40] We have seen that the progression by fifths generates all the 12 pitches. Does the same happen for the progression by fourth? or by major thirds? or by minor thirds? or by major seconds? or by minor seconds?

beats = battimenti,
chord = accordo,
equal temperament (or equal tuning) = temperamento equabile,
flat = bemolle (= nota più bassa di un semitono),
grand piano = pianoforte a coda,
half tone = semitone = semitono,
key = tasto, oppure chiave (ovvero tonalità di un brano),
musical score (or sheet music) = spartito (o partitura musicale),
note = nota,
octave = ottava,
overtone = (suono) armonico successivo a quello fondamentale,
pitch = altezza (di un suono),
pitch class (by octave reduction) = nota,
to play = suonare,
sharp = diesis (= nota più alta di un semitono),
staff = stave = pentagramma,
tone = nota, oppure somma di due semitoni,
tune = intonazione,
to tune = accordare,
tuning fork = diapason,
triad = triade (accordo di tre note).

References. A huge literature is devoted to musical theory.
In Italian language for instance there are the introductory texts

M. Fulgoni, A Sorrento: *Manuale di teoria musicale.* vol. 1,2. Edizioni musicali "la Nota", 2002, 2005

A. Frova: *Fisica nella musica.* Zanichelli, Bologna 1999

Ziegenrücker: *ABC della musica.* Rugginenti, Torino 2000

and the even more elementary

Shanet: *Come imparare a leggere la musica.* Rizzoli, Milano 2014.

The following article is also of interest:

S. Isola: *Su alcuni rapporti tra matematica e scale musicali.* Matematica, Cultura e Società. Aprile 2016, 31–49

S. Isola: *Su alcuni rapporti tra matematica e scale musicali.* Matematica, Cultura e Società. Aprile 2016, 31–49

Much is available on the net, too. E.g., S. Isola:

`https://mat.unicam.it/sites/mat.unicam.it/files/pls/Materiale_08_09/isola.pdf`

P. Oddifreddi:

`https://scuola.repubblica.it/blog/video/odifreddi-la-musica-spiegata-con-la-matematica/`

D. Kung:

`https://www.youtube.com/watch?v=ROeBLYvTv8Y`

L. Bernstein's:

`https://www.youtube.com/watch?v=8fHi36dvTdE`

2.5 Musical Questionnaire.

— What is a pure tone? What is a complex tone? What is a harmonic?
— What is the frequency of a sound? In which units is it measured?
— What is the timbre? Can it be related to the Fourier series of a sound?
— What is the difference between acoustic and psychoacoustic?
— What is the acoustic power of a sound? Is it a perceptual entity?
— What is loudness? Can it be measured?
— What is consonance? Is it an acoustic quality? Can beats be related to consonance?
— What does the Fechner law state?
— What is the octave equivalence? What is the octave reduction?
— What does it mean to transpose a piece of music?
— What is the diatonic scale? What is the chromatic scale?
— What is the harmonic series?
— What are the first 6 pitches of the harmonic series of F? and those of the harmonic series of C?
— Which problem there is for the seventh harmonic?
— The frequency of A_4 is conventionally fixed at 440 Hz. What are then the frequencies of the first 4 terms of the harmonic series of A_1? and of A_5?
— What is a musical interval?
— Can you transpose (forward) the scale of C major of a semitone?
— What is Just Intonation?
— What is the key of the scale of $F^\sharp$?
— Which note is the perfect fifth of C_2? and of $G_3\sharp$?
— $B^\sharp$ =?, $C^\flat$ =?;
— Is C a chromatic note? is $C^\sharp$ diatonic?
— Which pitch class is the major third of D? and the minor third of $F^\sharp$?
— May a diatonic scale include altered notes (i.e., flats or sharps)?
— What are the pitch classes of the G major scale? and of the C minor scale?
— What are two parallel scales?
— Is there any relation between the C minor scale and the E major scale?
and between the C minor scale and the $E^\flat$ major scale?
and between the $C^\sharp$ minor scale and the E major scale?
— Do the C major scale and the E minor scale consist of the same notes?
— Which is the relative scale of E major? and of E minor?
— Which is the parallel scale of E major? and that of E minor?
— What are the modes?
— What is a minor third? What is the minor third of C? and that of G?
— What are the first 3 terms of the progression of the fifths, starting from C?
— What are the drawbacks of the Pythagorean tuning?
— What is equal temperament?
— In equal temperament what is the frequency ratio of two notes at the distance of a semitone? and if the distance is a minot third?
— Does octave reduction entail Just Intonation?
— Is Just Intonation a theorem or an assumption?
— Does octave reduction entail equal temperament?
— Is equal temperament a theorem or an assumption?
— What are the fundamental differences between a harmonic series and a scale?
— We have seen that the progression by the fifths generates all the 12 pitch classes. Does the same happen for the progression of the fourths? or of the major thirds? or of the minor thirds? or of the major seconds? or of the minor seconds?

— (a) Let X be the family of all major and minor diatonic scales. What is the cardinality of X?

(b) Let us say that two elements of X are equivalent if they consist of the same notes. (For instance, the C major scale and the C minor scale are not equivalent.) How many are then the elements of the quotient set?

— What is the frequency ratio of an ascending harmonic major fourth? and that of a descending major fourth?

— What is the frequency ratio of an ascending harmonic major third? and that of a descending major third?

— What are the frequencies of the first four overtones of 1000Hz?

— Determine $m, n, p \in \mathbf{N}$ such that $\frac{\nu(C)}{m} = \frac{\nu(E)}{n} = \frac{\nu(G)}{p}$ in Just Tuning.

— Which ratios corresponds to the intervals $C_4 - G_4$, $G - 3 - C_4$ and $C_4 - G_5$ in Pythagorean tuning? and in Just Tuning?

— Assume that the first terms of the harmonics series have coefficients $a_0 = 0$, $a_1 = b_1 = 4$, $a_2 = b_2 = 3$, $a_3 = b_3 = 2$ $a_3 = b_3 = 1$, $a_n = b_n = 0$ for any $n > 4$. Establish the distribution of acoustic energy among the harmonics.