These pages are in progress. They contain:
— an abstract of the classes;
— notes on some (few) specific issues.

These notes are far from providing a full account of the classes, that are mainly based on the book of Renardy and Rogers ([ReRo]) for the first part of the course, and on some other notes for the second part.

1 Basic Second-Order PDEs

Classes. (see [ReRo; chap. 1]) Review of ordinary differential equations (ODEs). Existence and uniqueness result. Gronwall's lemma.

Laplace equation. Boundary conditions. Solution by separation of variables. Energy equality. Variational formulation. Minimizational formulation. Maximum principle.

Heat equation. Boundary and initial conditions. Solution by separation of variables. Backward heat equation. Duhem principle. Variational formulation. Energy inequality. Maximum principle.

Wave equation. Boundary and initial conditions. Solution by separation of variables. D'Alembert solution. Domain of dependence and domain of influence. Variational formulation. Energy conservation.

Schrödinger equation. Boundary and initial conditions. Solution by separation of variables. Variational formulation. Energy conservation.

Wave number and frequency of a harmonic wave. Dissipative and dispersive equations. Comparison of the qualitative properties of the heat, wave and Schrödinger equations.

Dispersion and Dissipation. A harmonic wave has the form

$$u(x,t) = \exp\{i(k \cdot x + ct)\}. \tag{1.1}$$

Here $k \in \mathbf{R}^N$ is the wave number; $c \in \mathbf{R}$, and c/k_i is the speed of the wave in the direction of x_i. Inserting a harmonic wave into a linear PDE $L(x,D)u = 0$, yields $L(x, i(k,c)) = 0$, which is named a *dispersion relation.* This is an algebraic equation relating k and c, and coincides with the vanishing of the symbol of L. If c depends on k, the equation is said *dispersive.* If the modulus of the wave is damped, the equation is said *dissipative.*

The heat equation is dissipative and not dispersive.

The Schrödinger equation is dispersive and not dissipative.

The wave equation is neither dissipative nor dispersive. [Evans 173]

An Example. Let us set $g(t) = \exp\{-1/t^2\}$ for any $t > 0$. The Tychonov function

$$u(x,t) := \sum_{n=0}^{\infty} g^{(n)}(t) \frac{x^{2k}}{(2k)!} \qquad \forall x \in \mathbf{R}, \forall t > 0 \tag{1.2}$$

provides a nontrivial solution for the Cauchy problem for the heat equation with homogeneous data. This function has a high order of growth at infinity, at variance with the solution constructed via the fundamental solution.

The heat semigroup [1] is continuous in $C^0_0(\mathbf{R}^N)$ (space of continuous functions that vanish at infinity), but not in $C^0_b(\mathbf{R}^N)$ (space of bounded and continuous functions).

[1] This is a notion that we have not yet introduced so far...

2 Characteristics and First-Order PDEs

Classes. (see [ReRo; chap. 2]) Multi-indices. Order of a PDE. Linear PD operators and principal part. Symbol of a linear PD operator and principal part.

Classification of 2^{nd}-order equations: elliptic, parabolic, hyperbolic. Characteristic (hyper)surfaces (or curves) for linear PD equations and systems. Cauchy-Kovalevskaya and Olmgren theorems (just statements). Classification of nonlinear PDEs of any order: semilinear, quasilinear, fully nonlinear equations.

Examples of linear first-order hyperbolic equations: homogeneous transport equation, transport equation with linear source, transport equation with known source; traffic equation [Sa; chap. 2].

Explicit solution of some first-order PDE.

Diagonalization of linear systems of first-order hyperbolic equations.

Remark on the Notion of Characteristic. In the literature there is some ambiguity in the use of the term "characteristic": in connection with PDEs, one deals with characteristic surface and characteristic curves.

Characteristic Surfaces. For any linear differential operator

$$L(x, D) = \sum_{|\alpha|\leq m} a_\alpha(x) D^\alpha \qquad (x \in \mathbf{R}^N), \tag{2.1}$$

$S = \{x \in \mathbf{R}^N : \varphi(x) = 0\}$ is called a *characteristic surface* (more precisely, a manifold of dimension $N-1$, thus a curve if $N = 2$) iff $\varphi \in C^1(\mathbf{R}^N)$, $D\varphi \neq 0$ everywhere, and if φ solves the following first-order PDE

$$\sum_{|\alpha|=m} a_\alpha(x)[D\varphi(x)]^\alpha = 0. \qquad \forall x \in \mathbf{R}^N. \tag{2.2}$$

(This is an example of a *Hamilton-Jacobi equation).*

Characteristic Curves. On the other hand, for first-order operators one often uses the term characteristic with a different meaning. First notice that (recalling that $D = (D_1, ..., D_N)$ and $\alpha = (\alpha_1, ..., \alpha_N)$)

$$(D\varphi)^\alpha = D^\alpha \varphi \qquad \forall \alpha \in \mathbf{R}^N \text{ such that } |\alpha| = 1. \tag{2.3}$$

Let us rewrite the principal part L_p of a first-order operator L in the form

$$L_p(x, D) = \sum_{j=1}^{N} a_j(x)\, D_j \qquad (x \in \mathbf{R}^N). \tag{2.4}$$

A curve of equation $x = \hat{x}(s)$ ($s \in\,]a, b[$) is often called a (projected) *characteristic curve* for this first-order operator iff

$$(a_1(\hat{x}(s)), ..., a_N(\hat{x}(s))) \quad \text{is parallel to} \quad \hat{x}'(s) \qquad \forall s \in\,]a, b[. \tag{2.5}$$

By rescaling the parameter s, we may assume that these two vector functions coincide. (Ahead we shall also encounter *unprojected characteristic curves).* Along this characteristic curve the (homogeneous) PDE is then reduced to

$$[L_p(x, D)u]_{x=\hat{x}(s)} = \sum_{j=1}^{N} \hat{x}_j'(s)\, D_j u(\hat{x}(s)) = \frac{d}{ds} u(\hat{x}(s)). \tag{2.6}$$

Therefore

> The characteristic curves are the curves along which
> any solution of the (homogeneous) first-order PDE is constant.

Therefore the Cauchy datum propagates along these curves. This provides a technique for constructing the solution of first-order PDEs, that is called the *method of characteristics.*

This multiple use of the term "characteristic" may be somehow confusing. However:
— for $N = 2$ the two notions coincide, and
— for $N > 2$ any characteristic manifold is a union of (disjoint) characteristic curves.

Integration of First-Order PDEs via the Method of Characteristics.
(a) Quasilinear PDEs. Let

$$V(x, y, u) = (a(x, y, u), b(x, y, u), c(x, y, u)) \text{ be continuous,} \tag{2.7}$$

and $u = u(x, y)$ be a solution (assumed to exist) of the quasilinear PDE

$$a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u) \quad \text{i.e.} \quad V \cdot (u_x, u_y, -1) = 0. \tag{2.8}$$

As the field $n = (u_x, u_y, -1)$ is normal to the surface S of equation $z = u(x, y)$, by (2.8) V is parallel to S. Any integral surface of the field V (namely, a surface that is tangent to the vector field at each point) is the graph of a solution of (2.8).

Any integral curve $(x(t), y(t), z(t))$ of the field V is named an (unprojected) characteristic curve; its projection on to the (x, y)-plane is indeed a projected characteristic curve. Any integral curve $(x(t), y(t), z(t))$ fulfills the nonparametric system

$$\frac{dx}{a} = \frac{dy}{b} = \frac{dz}{c} \qquad \text{i.e.} \qquad bdx = ady, cdx = adz, \tag{2.9}$$

or also, in parametric form,

$$x'(t) = a, \quad y'(t) = b, \quad z'(t) = c. \tag{2.10}$$

Any integral surface of the field V is a union of integral curves.

A curve Γ is said noncharacteristic for the vector field V iff it is not tangent to V at any point. In this case V has a unique integral surface that contains Γ. Provided that $a, b, c \in C^1$, any noncharacteristic datum thus defines a unique solution of the Cauchy problem that is determined by the prescription of the curve Γ.....

This method may easily be extended to equations in more than two independent variables.

(b) Semilinear PDEs. The equation (2.8) is a semilinear PDE if $a = a(x, y)$ and $b = b(x, y)$. In this case the resolution of the first two characteristic equations (that determine the *projected characteristics,* namely, the projection of the characteristics onto the x, y plane) is uncoupled from the third one.

The curve $\Gamma_0 = \{(f(s), g(s)) : s \in \mathbf{R}\}$ is not characteristic at any point if

$$\det \begin{pmatrix} a(f(s), g(s)) & b(f(s), g(s)) \\ f'(s) & g'(s) \end{pmatrix} \neq 0 \qquad \forall s \in \mathbf{R}, \tag{2.11}$$

i.e., $f'(s)b(f(s), g(s)) \neq g'(s)a(f(s), g(s))$ for any $s \in \mathbf{R}$.

Example 1. Let us consider the semilinear problem

$$u_x + 2u_y = u^2, \qquad u(x, 0) = h(x). \tag{2.12}$$

Here $\Gamma = \{(s, 0, h(s)) : s \in \mathbf{R}\}$. The characteristic system reads

$$x' = 1, \qquad y' = 2, \qquad z' = z^2. \tag{2.13}$$

Prescribing

$$x(s, 0) = s, \qquad y(s, 0) = 0, \qquad z(s, 0) = h(s), \tag{2.14}$$

we get the solution of the Cauchy problem (2.13), (2.14) (in which s is a parameter)

$$x(s,t) = t + s, \qquad y(s,t) = 2t, \qquad z(s,t) = h(s)/[1 - th(s)]. \tag{2.15}$$

By (2.12), we may eliminate (t,s), getting

$$u(x,y) = \frac{h(x - \frac{y}{2})}{1 - \frac{y}{2}h(x - \frac{y}{2})} \; . \tag{2.16}$$

(The equations (2.13) for x' and y' are linear, but that for z' is not so; this explains the onset of a singularity in the solution in finite time.)

Example 2. Let us consider the quasilinear problem

$$uu_x + yu_y = x, \qquad u(x,1) = 2x. \tag{2.17}$$

Here $\Gamma = \{(s,1,2s) : s \in \mathbf{R}\}$. The characteristic system reads

$$x' = z, \qquad y' = y, \qquad z' = x. \tag{2.18}$$

Prescribing

$$x(s,0) = s, \qquad y(s,0) = 1, \qquad z(s,0) = 2s, \tag{2.19}$$

we get

$$x(s,t) = \frac{3}{2}se^t - \frac{1}{2}se^{-t}, \qquad y(s,t) = e^t, \qquad z(s,t) = \frac{3}{2}se^t + \frac{1}{2}se^{-t}. \tag{2.20}$$

As a condition analogous to (2.11) holds, we may eliminate (t,s), getting

$$u(x,y) = x\frac{3y^2+1}{3y^2-1} \; . \tag{2.21}$$

Lagrange's Method of "Integrals" of the Equation. If $z = u(x,y)$ solves the equation (2.8), then, setting $\varphi(x,y,z) := u(x,y) - z$, we have

$$a(x,y,z)\varphi_x + b(x,y,z)\varphi_y + c(x,y,z)\varphi_z = 0. \tag{2.22}$$

Any function φ that fulfills the latter equation is named a "first integral" of the equation.

Let $\varphi_i = \varphi_i(x,y,z)$ $(i = 1,2)$ be two integrals, that are mutually independent — that is such that $\nabla\varphi_1$ is not parallel to $\nabla\varphi_2$ at any points. Let $F : \mathbf{R}^2 \to \mathbf{R}$ be any function of class C^1 such that $\nabla F \neq 0$ everywhere, and set $\theta(x,y,z) := F(\varphi_1(x,y,z), \varphi_2(x,y,z))$. Then

$$a(x,y,z)\theta_x + b(x,y,z)\theta_y + c(x,y,z)\theta_z = 0; \tag{2.23}$$

the equation

$$F(\varphi_1(x,y,z), \varphi_2(x,y,z)) = 0 \tag{2.24}$$

thus implicitly defines a solution $z = u(x,y)$ of the equation (2.8). In fact, for any $(k_1,k_2) \in \mathbf{R}^2$, the system $\varphi_1(x,y,z) = k_1$, $\varphi_2(x,y,z) = k_2$ defines a characteristic curve.

Linear System of First-Order PDEs. Let us consider the linear system

$$u_t + A(x,t)\cdot u_x = B(x,t)\cdot u + C(x,t) \qquad \text{for } (x,t) \in \mathbf{R}\times\mathbf{R}^+, \tag{2.25}$$

with $u : \mathbf{R}\times\mathbf{R}^+ \to \mathbf{R}^N$, and with A, B, C known (continuous) matrix-valued functions.

This system is *hyperbolic* at (x,t) iff $A(x,t)$ has a basis of (real) eigenfunctions;

it is said *strictly hyperbolic* at (x,t) iff $A(x,t)$ has N distinct real eigenvalues

$$\lambda_1(x,t) < ... < \lambda_N(x,t) \tag{2.26}$$

(and then a basis of eigenvectors $y_1(x,t), ..., y_N(x,t)$). Let us denote by Λ the matrix whose columns are these eigenvectors. For any (x,t), (2.25) may then be diagonalized; that is, via the linear transformation $\mathbf{R}^N \to \mathbf{R}^N : u \mapsto v = \Lambda \cdot u$ (and $B \mapsto \tilde{B}$, $C \mapsto \tilde{C}$), it may be reduced to a system of N independent equations:

$$(v_k)_t + \lambda_k(x,t)(v_k)_x = \tilde{B}(x,t)\cdot v_k + \tilde{C}(x,t) \qquad k = 1,..,N. \tag{2.27}$$

This corresponds to the following equations for the characteristic curves $x = \hat{x}(t)$:

$$\hat{x}_k'(t) = \lambda_k(\hat{x}(t),t) \qquad k = 1,..,N. \tag{2.28}$$

In particular, if A is constant then the $\lambda_1, ..., \lambda_N$ are also constant, so that there exist $c_1, ..., c_N \in \mathbf{R}$ such that

$$\hat{x}_k(t) = \lambda_k t + c_k \qquad k = 1,..,N. \tag{2.29}$$

If $B = C = 0$ then $\tilde{B} = \tilde{C} = 0$, and the solutions $v_1(x,t), ..., v_N(x,t)$ are then travelling waves:

$$v_k(x,t) = f_k(x - \lambda_k t) \qquad k = 1,..,N. \tag{2.30}$$

This is easily extended to semilinear equations (i.e., with a second member that is nonlinear in u).

3 Conservation Laws

Classes. (see [ReRo; chap. 3]) Examples of nonlinear first-order hyperbolic equations and systems. Characteristics. Bases of left and right eigenvectors for the matrix of systems of conservation laws. Riemann invariants.

Cauchy problem for a single conservation law. Determination of the strong solution via the method of characteristics. Possible failure in finite time. Weak formulation and Rankine-Hugoniot condition. Rarefaction wave and shocks waves. Lax entropy condition. Physical and nonphysical shocks. Riemann problem.

Entropy and entropy-flux pairs; entropy criterion. "Entropy-jump condition". Viscosity solutions.

Traffic equation. Wave velocity. Rarefaction wave at a traffic light. Car crash. (see [Salsa; chap. 4])

Conservation Laws. Let us consider a system of *conservation laws* of the form

$$D_t u + D_x F(u) = 0 \qquad \text{in } \mathbf{R}\times\mathbf{R}^+, \tag{3.1}$$

that is,

$$\textstyle\sum_j \delta_{ij} D_t u_j + \sum_j D_{u_j} F_i(u) D_x u_j = 0 \qquad \text{in } \mathbf{R}\times\mathbf{R}^+, \text{ for } i = 1,..,N, \tag{3.2}$$

with $u : \mathbf{R}^2 \to \mathbf{R}^M$, $F : \mathbf{R}^M \to \mathbf{R}^M$. Let us set $A(u) = \nabla F(u)$ for any $u \in \mathbf{R}^M$. Let $\Gamma = \{(x,t) : \varphi(x,t) = 0\}$, with $\varphi \in C^1(D)$ such that $\nabla\varphi \neq 0$ everywhere; this curve is characteristic iff

$$\text{Det}[I D_t\varphi(x,t) + A(u(x,t))\cdot D_x\varphi(x,t)] = 0 \qquad \forall (x,t) \in \Gamma. \tag{3.3}$$

If in a neighbourhood of some point Γ may be represented parametrically as $]a,b[\to \mathbf{R} : t \mapsto (\hat{x}(t), t)$, then we may take $\varphi(x,t) = \hat{x}(t) - x$. (3.3) then yields

$$\text{Det}[I\hat{x}'(t) - A(u(\hat{x}(t),t))] = 0 \qquad \forall t \in]a,b[. \tag{3.4}$$

The system (3.3) is said *hyperbolic* at some $u \in D$ iff the matrix $A(u)$ has a basis of eigenvectors, and *strictly hyperbolic* iff it has N distinct real eigenvalues: $\lambda_1(u) < ... < \lambda_N(u)$. Each of these eigenvalues is associated to a left and a right eigenvector: $\ell_k(u), r_k(u)$ for $k = 1, ..., N$. (Notice that $\ell_k(u)$ is orthogonal to $r_h(u)$ if $k \neq h$.)

If the system is hyperbolic at all points, then $A(u)$ may be diagonalized, so that (3.1) is reduced to N uncoupled equations. This yields N characteristic curves $(\hat{x}_k(t), t)$, such that

$$\hat{x}_k'(t) = \lambda_k(u(\hat{x}_k(t), t)) \qquad \forall t \in]a, b[\ (k = 1, ..., N). \tag{3.5}$$

Riemann Invariants. For any $k \in \{1, ..., M\}$, a smooth function $w : D \to \mathbf{R}$ is said a *k-Riemann invariant* iff

$$r_k(u) \cdot \nabla w(u) = 0 \qquad \forall u \in \mathbf{R}^M. \tag{3.6}$$

If u is a solution of class C^1 of the system (3.1), then a k-Riemann invariant ($k \in \{1, ..., N\}$) is constant along the characteristic curve of equation $(\hat{x}_k(t), t)$, namely the curve that fulfills (3.5) for the same k. This entails that, if a system has N Riemann invariants whose gradients are linearly independent, then it may be diagonalized. This hypothesis is fulfilled by all strictly hyperbolic systems of two conservation laws, but not by more general systems.

On the Derivation of the Rankine-Hugoniot Condition. [Salsa] Let us consider the single scalar conservation law

$$D_t u + D_x f(u) = 0 \qquad \text{in } \mathbf{R} \times \mathbf{R}^+, \tag{3.7}$$

with $f \in C^2(\mathbf{R}^2)$ and (e.g.) $f''(u) > 0$ for any u. Let u be a solution, and $x = \hat{x}_k(t)$ represent a shock, namely a regular curve along which u has a jump; let us assume that $u \in C^1$ outside the shock curve. Let us fix any $t \in]a, b[$, assume that $x_1 < \hat{x}_k(t) < x_2$, and integrate (3.7) in $]x_1, x_2[$. This yields

$$D_t \int_{x_1}^{x_2} u\, dx + f(u(x_2, t)) - f(u(x_1, t)) = 0. \tag{3.8}$$

(This is formally equivalent to (3.7), but it may be written also if u is just continuous.) Notice that

$$\begin{aligned} &D_t \textstyle\int_{x_1}^{x_2} u\, dx = D_t \int_{x_1}^{\hat{x}(t)} u\, dx + D_t \int_{\hat{x}(t)}^{x_2} u\, dx \\ &= \textstyle\int_{x_1}^{\hat{x}(t)} D_t u\, dx + \int_{\hat{x}(t)}^{x_2} D_t u\, dx + \hat{x}'(t) u(\hat{x}(t) - 0, t) - \hat{x}'(t) u(\hat{x}(t) + 0, t), \end{aligned} \tag{3.9}$$

so that (3.7) also reads

$$\begin{aligned} &\textstyle\int_{x_1}^{\hat{x}(t)} D_t u\, dx + \int_{\hat{x}(t)}^{x_2} D_t u\, dx + \hat{x}'(t) u(\hat{x}(t) - 0, t) - \hat{x}'(t) u(\hat{x}(t) + 0, t) \\ &+ f(u(x_2, t)) - f(u(x_1, t)) = 0. \end{aligned} \tag{3.10}$$

Passing to the limit as $x_1 \to \hat{x}(t) - 0$ and $x_2 \to \hat{x}(t) + 0$, we get

$$\hat{x}'(t)[u(\hat{x}(t) - 0, t) - u(\hat{x}(t) + 0, t)] + f(u(\hat{x}(t) + 0, t)) - f(u(\hat{x}(t) - 0, t)) = 0, \tag{3.11}$$

that is, the Rankine-Hugoniot condition:

$$\hat{x}'(t) = [[f(u)]]/[[u]] \qquad ([[u]] := \text{ jump of } u \text{ across the shock curve}). \tag{3.12}$$

4 Maximum Principle

Classes. (see [ReRo; chap. 4]) Linear elliptic operators of the second order in nondivergence and divergence forms. Weak maximum principle for linear elliptic operators in non-divergence form. Hopf's strong maximum principle. Maximum principle for linear parabolic operators of the second order in non-divergence form.

Integral formulation of linear elliptic and parabolic operators of the second order in divergence form, and derivation of the maximum principle.

Weak Maximum Principle for Elliptic Operators. Let Ω be a bounded Euclidean domain and $x_0 \in \partial\Omega$ be such that there exists a ball contained in $\bar{\Omega}$ that is tangent to $\partial\Omega$ at x_0. Let us define two linear elliptic operators of the second order (in non-divergence form):

$$\begin{aligned} &L_0 u := a_{ij}(x) D_i D_j u + b_i(x) D_i u, \quad Lu := L_0 u + c(x) u \\ &\text{with } a_{ij}, b_i, c \in C^0(\Omega) \ \ \forall i,j, \text{ and } A = \{a_{ij}\} \text{ everywhere positive definite.} \end{aligned} \tag{4.1}$$

(For instance, $L_0 = \Delta$, $Lu := \Delta + cI$.) We shall denote by ν the outward-oriented unit normal vector field on $\partial\Omega$, and by $\partial u(x_0)/\partial\nu$ the corresponding normal derivative. We shall assume that $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$.

The weak maximum principle states that

$$-L_0 u \le 0 \quad \text{in } \Omega \quad \Rightarrow \quad \max_{\bar{\Omega}} u \le \max_{\partial\Omega} u; \tag{4.2}$$

this entails that

$$-Lu \le 0, \quad c \le 0 \quad \text{in } \Omega \quad \Rightarrow \quad \max_{\bar{\Omega}} u \le \max_{\partial\Omega} u^+. \tag{4.3}$$

Let $f \in C^0(\Omega)$ and $g \in C^0(\partial\Omega)$ be prescribed, and consider the *boundary value problem*

$$-Lu = f \quad \text{in } \Omega, \qquad u = g \quad \text{on } \partial\Omega. \tag{4.4}$$

(This is a nonhomogeneous Dirichlet problem.) Still assuming that $c \le 0$ in Ω, (4.3) entails the *monotone dependence on the data:*

$$\begin{aligned} -Lu_1 \le -Lu_2 \quad & \text{in } \Omega \\ u_1 \le u_1 \quad & \text{on } \partial\Omega \end{aligned} \quad \Rightarrow \quad u_1 \le u_2 \quad \text{in } \Omega. \tag{4.5}$$

In turn this yields the uniqueness of the solution.

A Remark of the Strong Maximum Principle. If $u \le u(x_0)$ in a neighborhood of x_0, then it is clear that $\partial u(x_0)/\partial\nu \ge 0$. Let U be a neighborhood of x_0. E. Hopf proved that

$$u(x) < u(x_0), \quad -L_0 u(x) \le 0 \quad \forall x \in U \cap \Omega \quad \Rightarrow \quad \partial u(x_0)/\partial\nu > 0. \tag{4.6}$$

(For $N = 1$ and $L_0 = D^2$ this is clearly seen.)

We remark that if $c \le 0$ in U and $u(x_0) > 0$, then, possibly restricting the neighborhood U of x_0, $cu \le 0$ in U. Therefore

$$-L_0 u := -Lu + cu \le -Lu \qquad \text{in } U \cap \Omega. \tag{4.7}$$

By (**??**) we then conclude that

$$\begin{aligned} &u(x_0) > 0, \quad u(x) \le u(x_0), \quad c(x) \le 0, \quad -Lu(x) \le 0 \quad \forall x \in U \cap \Omega \\ &\qquad \Rightarrow \quad \partial u(x_0)/\partial\nu > 0. \end{aligned} \tag{4.8}$$

(4.6) and (4.8) respectively provide the strong maximum principle in the following reduced forms:

(a) If $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ is not constant and $-L_0 u \le 0$ in Ω, then u cannot attain its maximum in Ω.

(b) If $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ is not constant, $c \le 0$ and $-Lu \le 0$ in Ω, then u cannot attain a positive maximum in Ω.

Weak Maximum Principle for Parabolic Operators. Let L_0 and L be as above, with $x \in \Omega$ replaced by $(x,t) \in \Omega \times]0,T[$; let us fix any $T > 0$, and set

$$\mathcal{L}_0 := L_0 - D_t, \quad \mathcal{L} := L - D_t \qquad \text{in } Q := \Omega \times]0,T[. \tag{4.9}$$

Let us also set $\Sigma := (\partial\Omega) \times]0,T]$, and define the *parabolic boundary* $\partial_p Q := (\Omega \times \{0\}) \cup \Sigma$.

The weak maximum principle states that

$$-\mathcal{L}_0 u \le 0 \quad \text{in } Q \quad \Rightarrow \quad \max_{\bar{Q}} u \le \max_{\partial_p Q} u; \tag{4.10}$$

this entails that

$$-\mathcal{L} u \le 0, \quad c \le 0 \quad \text{in } Q \quad \Rightarrow \quad \max_{\bar{Q}} u \le \max_{\partial_p Q} u^+. \tag{4.11}$$

Let $f \in C^0(Q)$, $u^0 \in C^0(\Omega)$ and $g \in C^0(\partial_p Q)$ be prescribed, and consider the *initial-boundary value problem*

$$-\mathcal{L} u = f \quad \text{in } Q, \qquad u = u^0 \quad \text{on } \Omega \times \{0\}, \qquad u = g \quad \text{on } \Sigma. \tag{4.12}$$

(This is a nonhomogeneous Cauchy-Dirichlet problem.) Still assuming that $c \le 0$ in Ω, (4.11) entails the *monotone dependence on the data:*

$$\begin{array}{ll} -\mathcal{L} u_1 \le -\mathcal{L} u_2 & \text{in } Q \\ u_1^0 \le u_2^0 & \text{in } \Omega \\ u_1 \le u_1 & \text{on } \Sigma \end{array} \quad \Rightarrow \quad u_1 \le u_2 \quad \text{in } Q. \tag{4.13}$$

In turn this yields the uniqueness of the solution.

A maximum principle may also be derived for parabolic equations. In this case one concludes that if u attains its maximum at some $(x_0,t_0) \notin \partial_p Q$, then $u(x,t) = u(x_0,t_0)$ for any $(x,t) \in \Omega \times [0,t_0]$.

By applying the weak/strong maximum principle to $-u$, analogous weak/strong minimum principles are easily derived, in both the elliptic and parabolic cases.

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