Laplace transform

Nota: Una trattazione elementare ma più ampia della transformazione di Laplace è offerta ad esempio dal primo capitolo di

M. Marini: *Metodi matematici per lo studio delle reti elettriche.* C.E.D.A.M., Padova, 1999.

La seguente trattazione è basata in parte sul testo

G. Gilardi: *Analisi tre.* McGraw-Hill, Milano 1994.

This chapter includes the following sections:

1 Laplace Transform of Functions

This transform is strictly related to that of Fourier, and like the latter it allows one to transform ODEs to algebraic equations. But the Laplace transform is especially suited for the study of initial value problems, whereas the Fourier transform is appropriate for problems on the whole real line. In applications the theory of Laplace transform is also called *symbolic* or *operational calculus.*

From Fourier to Laplace transform. Let us first present some informal remarks. Let us denote by $\widehat{u}$ the Fourier transform of a transformable function $u : \mathbf{R} \to \mathbf{C}$:

$$\widehat{u}(\xi) := \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} u(t) e^{-i\xi t}\, dt \qquad \text{for } \xi \in \mathbf{R}. \tag{1.1}$$

The inversion formula is one of the main elements of interest of this transform:

$$u(t) := \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} \widehat{u}(\xi) e^{i\xi t}\, d\xi \qquad \text{for } t \in \mathbf{R}. \tag{1.2}$$

This represents u as an (integral) average of the periodic functions $w_\xi : \mathbf{R} \to \mathbf{C} : t \mapsto e^{i\xi t}$ parameterized by $\xi \in \mathbf{R}$ and with weight $\widehat{u}(\xi)$. Each of the functions w_ξ is called a *harmonic* with frequency ξ, [1] as $w = w_\xi$ solves the equation of harmonic motion: [2]

$$w''(t) + \xi^2 w(t) = 0 \qquad \text{for } t \in \mathbf{R}. \tag{1.3}$$

Because of the classical method of *variation of the constants,* the solutions of this equation can be used for the study of the nonhomogeneous equation $w''(t) + \xi^2 w(t) = f(t)$ for $t \in \mathbf{R}$, for a prescribed function f. (More specifically, one may look for a solution of the form $w = zw_\xi$, with w_ξ solution of (1.3) and $z = z(t)$ to be determined. By replacing this expression in (1.3), we get the new equation $z''w + 2z'w' = f$, which is easily integrated.)

[1] ξ represents the *angular frequency,* (also named *pulsation),* namely the ratio radians/time. Therefore $\xi/(2\pi)$ is the frequency, namely the ratio cycles/time.

[2] Despite of (1.3), the functions w_ξ cannot be regarded as eigenfunctions of the second derivative in either L^1 or L^2, since they are not elements of either space.

As $|w_\xi(t)| \le 1$ for any $\xi, t \in \mathbf{R}$, the Fourier integral (1.2) converges for any $u \in L^1$. Anyway we have seen that one may also assume $u \in L^2$, provided that the Fourier integral is understood in the sense of the principal value of Cauchy; and the domain of this transform can be further extended.

For any $\eta \in \mathbf{R}$, let us also consider another equation of great interest:

$$w'(t) + \eta w(t) = 0 \qquad \text{for } t \in \mathbf{R}. \tag{1.4}$$

The solutions of this equation are proportional to $w_\eta : \mathbf{R} \to \mathbf{R} : t \mapsto e^{-\eta t}$; if $\eta \ne 0$, this function diverges exponentially as $t \to +\infty$ or $t \to -\infty$, depending on the sign of η. Analogously to (1.3), the classical method of variation of the constants allows one to construct a solution for the nonhomogeneous equations $w'(t) + \eta w(t) = f(t)$. [3]

In analogy with (1.1)), for the study of the equation $w'(t) + \eta w(t) = f(t)$ first we set

$$\widetilde{u}(\eta) := \int_{\mathbf{R}} u(t) e^{-\eta t}\, dt \qquad \text{for } \eta \in \mathbf{R}. \tag{1.5}$$

For any $\eta > 0$ ($\eta < 0$, respect.) this integral converges only if $|u(t)|$ exponentially decays to 0 as $t \to -\infty$ (as $t \to +\infty$, respect.). Anyway in many cases of applicative interest the equation (1.4) is not set on the whole $\mathbf{R}$, but just on $\mathbf{R}^+$; in those cases it is convenient to restrict oneself to functions that vanish for any $t < 0$. [4]

More generally, we can replace the real variable η by a complex variable s, since $|e^{-st}| = e^{-\operatorname{Re}(s)t}$ for any $t \in \mathbf{R}$ (time remains real!). We can then consider the kernels $\mathbf{R} \to \mathbf{C} : t \mapsto e^{-st}$, parameterized by $s \in \mathbf{C}$. For imaginary s we retrieve periodic functions, whereas for real s these functions either grow or decay exponentially. [5] Let us then define the Laplace transform

$$u \mapsto \widetilde{u}(s) := \int_{\mathbf{R}} u(t) e^{-st}\, dt \qquad \text{for } s \in \mathbf{C}, \tag{1.6}$$

which includes (1.5) for s real and the Fourier transform for s imaginary (the conventional factor $1/\sqrt{2\pi}$ apart). Concerning the convergence of this integral, what we said for (1.5) holds here also, and we shall confine ourselves to causal signals.

Setting $s = x + iy$, we thus have

$$U_x(y) := \widetilde{u}(x + iy) = \int_{\mathbf{R}} u(t) e^{-xt} e^{-iyt}\, dt \qquad \text{for } x, y \in \mathbf{R}. \tag{1.7}$$

The variable s is often referred to as a (complex) frequency, although just its imaginary part really represents a frequency, cf. (1.2). As U_x is the Fourier transform of the function $t \mapsto \sqrt{2\pi}\, u(t) e^{-xt}$, (1.6) may be regarded as the Fourier transform of an array of inputs parameterized by $x \in \mathbf{R}$.

The right side of (1.7) has a meaning whenever $e^{-xt} u(t) \in L^1$; if u is causal, the larger is x the less restrictive is this condition. This allows one to apply this formulation to the Laplace transform of a wide family of causal functions of L^1_{loc}. [6]

[3] Just as above, despite of (1.4), the functions w_η cannot be regarded as eigenfunctions of the first derivative in either L^1 or L^2, since they are not elements of either space.

[4] These are called *causal signals* in *signal analysis.* We shall occasionally refer to the terminology of this field of engineering.

[5] The functions $\widetilde{u}(\eta)$ cannot be regarded as eigenfunctions of the first derivative in $\mathcal{S}'$, since they do not belong to that space.

[6] There is thus little interest to extend this transform to L^2.

Functional set-up. The Laplace transform deals with functions of a single real variable, which typically represents time and is denoted by t. On the basis of the previous remarks, let us define the class $D_{\mathcal{L}}$ of the *transformable functions,* and for any $u \in D_{\mathcal{L}}$ the *abscissa of (absolute) convergence* $\lambda(u)$, the *convergence half-plane* $\mathbf{C}_{\lambda(u)}$, and finally the *Laplace transform* $\mathcal{L}(u)$: [7]

$$D_{\mathcal{L}} := \left\{u \in L^1_{\text{loc}} : u(t) = 0 \ \ q\forall t < 0, \exists x \in \mathbf{R} : e^{-xt}u(t) \in L^1_t\right\}, \tag{1.8}$$

$$\lambda(u) := \inf \left\{x \in \mathbf{R} : e^{-xt}u(t) \in L^1_t\right\} \ (\in [-\infty, +\infty[) \qquad \forall u \in D_{\mathcal{L}}, \tag{1.9}$$

$$\mathbf{C}_{\lambda(u)} := \{s \in \mathbf{C} : \ \text{Re}(s) > \lambda(u)\} \qquad \forall u \in D_{\mathcal{L}}, \tag{1.10}$$

$$[\mathcal{L}(u)](s) := \int_{\mathbf{R}} e^{-st}u(t)\,dt \qquad \forall s \in \mathbf{C}_{\lambda(u)}, \forall u \in D_{\mathcal{L}}. \tag{1.11}$$

The transformed function $\mathcal{L}(u)$ is also called the *Laplace integral.* $D_{\mathcal{L}}$ is the set of locally integrable causal signals that have at most exponential growth. This is a linear space, and the Laplace transform is obviously linear: for any $u, v \in D_{\mathcal{L}}$ ed ogni $\mu_1, \mu_2 \in \mathbf{C}$,

$$\lambda(\mu_1 u + \mu_2 v) = \max\{\lambda(u), \lambda(v)\}, \quad \mathcal{L}(\mu_1 u + \mu_2 v) = \mu_1 \mathcal{L}(u) + \mu_2 \mathcal{L}(v). \tag{1.12}$$

The Lebesgue integral (1.11) converges for any $s \in \mathbf{C}_{\lambda(u)}$. For some functions this integral may converge also for some complex s with $\text{Re}(s) = \lambda(u)$. But the Laplace transformed function $\mathcal{L}(u)$ is defined just in $\mathbf{C}_{\lambda(u)}$, since some properties might fail if $\text{Re}(s) \leq \lambda(u)$.

Although we called $\mathbf{C}_{\lambda(u)}$ the convergence *half-plane,* w We do not exclude $\mathbf{C}_{\lambda(u)} = \mathbf{C}$ (which we still call the convergence *half-plane).* Indeed $\lambda(u) = -\infty$ if the transformable functions decays more than exponentially; in particular this occurs for any compactly supported function of $D_{\mathcal{L}}$. On the other hand, we exclude $\mathbf{C}_{\lambda(u)} = \emptyset$, i.e. $\lambda(u) = +\infty$.

In the definition of the elements of $D_{\mathcal{L}}$ we shall often encounter the *Heaviside function* H (also called *unit step):*

$$H(t) := 0 \qquad \forall t \leq 0, \qquad H(t) := 1 \qquad \forall t > 0.$$

The occurrence of the factor $H(t)$ will guarantee causality. For instance, it is easily checked that

$$\begin{aligned} &H(t) \in D_{\mathcal{L}}, \quad \lambda(H) = 0, \\ &t^\alpha H(t) \in D_{\mathcal{L}}, \quad \lambda(t^\alpha H(t)) = 0 \ \Leftrightarrow \ \text{Re}(\alpha) > -1; \end{aligned} \tag{1.13}$$

$$e^{-t^2}H(t) \in D_{\mathcal{L}}, \quad \lambda(e^{-t^2}H(t)) = -\infty; \qquad e^{t^2}H(t), t^{-1}H(t) \notin D_{\mathcal{L}}. \tag{1.14}$$

The next result relates the Fourier and Laplace transforms.

Theorem 1.1. *For any* $u \in D_{\mathcal{L}}$,

$$[\mathcal{L}(u)](x + iy) = \sqrt{2\pi}\left[\mathcal{F}(e^{-xt}u(t))\right](y) \qquad \forall y \in \mathbf{R}, \forall x > \lambda(u). \tag{1.15}$$

[7] We write L^1_t in order to make clear that t is the integration variable.

Here we speak of abscissa of *absolute* convergence since the Lebesgue integral is absolutely convergent. Some authors define the Laplace transform as an improper integral, rather than as a Lebesgue integral. Accordingly, they do not prescribe absolute convergence; this has some analogy with what we shall do ahead extending the Laplace transform to distributions. This has some consequences on some properties of the transform.

What we introduced is called the *unilateral* Laplace transform, as the domain of convergence is a half-plane. In literature a *bilateral* Laplace transform is also defined: in this case u need not be causal. The domain of convergence is then a strip of the form $\{s \in \mathbf{C} : \lambda_1(u) < \text{Re}(s) < \lambda_2(u)\}$, with $-\infty \leq \lambda_1(u) < \lambda_2(u) \leq +\infty$. This bilateral transform is much less used than the unilateral one.

Vice versa, $u \in D_{\mathcal{L}}$ *and* $\lambda(u) \le 0$ *for any causal* $u \in L^1$*. If* $u \in L^1$ *and* $\lambda(u) < 0$ *then* [8]

$$[\mathcal{F}(u)](y) = \frac{1}{\sqrt{2\pi}}[\mathcal{L}(u)](iy) \qquad \forall y \in \mathbf{R}. \tag{1.16}$$

By the foregoing result, several properties of the Fourier transform are easily extended to the Laplace transform. This also applies to the next statement.

Proposition 1.2. *For any* $u \in D_{\mathcal{L}}$*,*

$$v(t) = u(t - t_0) \quad \Rightarrow \quad \lambda(v) = \lambda(u), \quad \widetilde{v}(s) = e^{-t_0 s}\,\widetilde{u}(s) \qquad \forall t_0 > 0, \tag{1.17}$$

$$v(t) = e^{s_0 t}u(t) \quad \Rightarrow \quad \lambda(v) = \lambda(u) + \mathrm{Re}(s_0), \quad \widetilde{v}(s) = \widetilde{u}(s - s_0) \qquad \forall s_0 \in \mathbf{C}, \tag{1.18}$$

$$v(t) = u(\omega t) \quad \Rightarrow \quad \lambda(v) = \omega\lambda(u), \quad \widetilde{v}(s) = \tfrac{1}{\omega}\widetilde{u}\Big(\tfrac{s}{\omega}\Big) \qquad \forall \omega > 0. \tag{1.19}$$

The assertions about convergence abscissas are easily checked: the delay does not modify the behaviour of the function for $t \to +\infty$; the exponential factor $e^{s_0 t}$ instead entails a translation of the convergence abscissa.

Examples. For any $u \in D_{\mathcal{L}}$,

$$u(t) = H(t) \quad \Rightarrow \quad \lambda(u) = 0, \quad \widetilde{u}(s) = \frac{1}{s}\ , \tag{1.20}$$

$$u(t) = e^{\gamma t}H(t)\ (\gamma \in \mathbf{C}) \quad \Rightarrow \quad \lambda(u) = \mathrm{Re}(\gamma), \quad \widetilde{u}(s) = \frac{1}{s - \gamma}\ , \tag{1.21}$$

$$u(t) = \cos(\omega t)H(t)\ (\omega \in \mathbf{R}) \quad \Rightarrow \quad \lambda(u) = 0, \quad \widetilde{u}(s) = \frac{s}{s^2 + \omega^2}\ , \tag{1.22}$$

$$u(t) = \sin(\omega t)H(t)\ (\omega \in \mathbf{R}) \quad \Rightarrow \quad \lambda(u) = 0, \quad \widetilde{u}(s) = \frac{w}{s^2 + \omega^2}\ , \tag{1.23}$$

$$u(t) = \cosh(\omega t)H(t)\ (\omega \in \mathbf{R}) \quad \Rightarrow \quad \lambda(u) = |\omega|, \quad \widetilde{u}(s) = \frac{s}{s^2 - \omega^2}\ , \tag{1.24}$$

$$u(t) = \sinh(\omega t)H(t)\ (\omega \in \mathbf{R}) \quad \Rightarrow \quad \lambda(u) = |\omega|, \quad \widetilde{u}(s) = \frac{w}{s^2 - \omega^2}\ , \tag{1.25}$$

$$u(t) = t^k H(t)\ (k \in \mathbf{N}) \quad \Rightarrow \quad \lambda(u) = 0, \quad \widetilde{u}(s) = \frac{k!}{s^{k+1}}\ . \tag{1.26}$$

The reader is asked to check these properties.

The final formula is generalized by

$$u(t) = t^a H(t)\ (a > -1) \quad \Rightarrow \quad \lambda(u) = 0, \quad \widetilde{u}(s) = \frac{\Gamma(a+1)}{s^{a+1}}, \tag{1.27}$$

since, by definition of the classical Euler function Γ,

$$\widetilde{u}(s) = \int_0^{+\infty} e^{-st}t^a\,dt = \frac{1}{s^{a+1}}\int_0^{+\infty} e^{-y}y^a\,dy =: \frac{\Gamma(a+1)}{s^{a+1}} \qquad \forall s \in \mathbf{C}_{\lambda(u)} = \mathbf{C}_0.)$$

Incidentally, notice that $1/s$ is definite for any complex $s \ne 0$, but $\widetilde{H}(s) = 1/s$ only if $\mathrm{Re}(s) > \lambda(H) = 0$.

[8] For $\lambda(u) = 0$ there is a formal inconvenience: if $y \in \mathbf{R}$, $[\mathcal{F}(u)](y)$ obviously exists as $u \in L^1$; but, as we saw, in this case it is not legitimate to write $[\mathcal{L}(u)](iy)$.

The Laplace transform operates on periodic functions in a different way from the Fourier transform, as it is shown by the next result. Notice that, because of causality, here u is assumed to be the restriction of a periodic function only for positive time.

Theorem 1.3. *(Periodic Functions) For any $T > 0$, let $u \in D_{\mathcal{L}}$, $u \not\equiv 0$ be such that $u(t+T) = u(t)$ for any $t > 0$. Then $\lambda(u) = 0$ and, setting*

$$w(t) := u(t) \quad \forall t \in [0,T], \qquad w(t) := 0 \quad \forall t \in \mathbf{R} \setminus [0,T], \tag{1.28}$$

we have

$$\widetilde{u}(s) = \frac{1}{1 - e^{-sT}} \widetilde{w}(s) \qquad \forall s \in \mathbf{C}_0. \tag{1.29}$$

Notice that $1 - e^{-sT} \neq 0$ for any $s \in \mathbf{C}_0$, and that the hypotheses entail that $u \in L^1(0,T)$, but not $u \in L^\infty(0,T)$.

Proof. For any $x > 0$, changing the integration variable, using the periodicity and setting $C := \int_0^T |w(\tau)|\, d\tau$ ($\neq 0$ as $u \not\equiv 0$), we have

$$\begin{aligned} \int_{\mathbf{R}} e^{-xt} |u(t)|\, dt &= \sum_{n=0}^{\infty} \int_{nT}^{(n+1)T} e^{-xt} |u(t)|\, dt = \sum_{n=0}^{\infty} e^{-nTx} \int_0^T e^{-x\tau} |w(\tau)|\, d\tau \\ (\text{as } e^{-x\tau} \leq 1) &\leq \sum_{n=0}^{\infty} e^{-nTx} \int_0^T |w(\tau)|\, d\tau = C \sum_{n=0}^{\infty} e^{-nTx} < +\infty. \end{aligned} \tag{1.30}$$

On the other hand, for any $x < 0$,

$$\begin{aligned} \int_{\mathbf{R}} e^{-xt} |u(t)|\, dt &= \sum_{n=0}^{\infty} \int_{nT}^{(n+1)T} e^{-xt} |u(t)|\, dt = \sum_{n=0}^{\infty} e^{-nTx} \int_0^T e^{-x\tau} |w(\tau)|\, d\tau \\ (\text{as } e^{-x\tau} \geq 1) &\geq \sum_{n=0}^{\infty} e^{-nTx} \int_0^T |w(t)|\, dt = C \sum_{n=0}^{\infty} e^{-nTx} = +\infty. \end{aligned} \tag{1.31}$$

It follows that $\lambda(u) = 0$. [9] Setting $u_T(t) := u(t-T)$ for any $t \in \mathbf{R}$ and noticing that $w = u - u_T$, we have

$$\widetilde{w}(s) = \widetilde{u}(s) - \widetilde{u_T}(s) = \widetilde{u}(s) - e^{-sT}\widetilde{u}(s) \stackrel{(1.17)}{=} (1 - e^{-sT})\widetilde{u}(s) \qquad \forall s \in \mathbf{C}_{\lambda(u)},$$

that is (1.29). □

Theorem 1.4. *(Convolution Theorem) For any $u, v \in D_{\mathcal{L}}$, $u * v \in D_{\mathcal{L}}$ and*

$$\lambda(u * v) \leq \max\{\lambda(u), \lambda(v)\}, \qquad \mathcal{L}(u * v) = \mathcal{L}(u)\, \mathcal{L}(v). \tag{1.32}$$

More generally, for any integer $N \geq 2$ and for any $u_1, ..., u_N \in D_{\mathcal{L}}$, we have $u_1 * ... * u_N \in D_{\mathcal{L}}$ and

$$\lambda(u_1 * ... * u_N) \leq \max\{\lambda(u_i) : i = 1, ..., N\}, \qquad \mathcal{L}(u_1 * ... * u_N) = \prod_{i=1}^{N} \mathcal{L}(u_i). \tag{1.33}$$

[9] More synthetically, one may notice that, as u is periodic and integrable in any interval of the form $]nT, (n+1)T[$, $e^{-xt}u(t) \in L^1$ for all $x > 0$ and for no $x < 0$. Therefore $\lambda(u) = 0$.

The inequality (1.32) may be strict: e.g. consider $u \equiv 0$ and any $v \in D_{\mathcal{L}}$.

The proof of (1.32) mimics that of the Fourier transform of the convolution, and is left to the reader. (1.33) follows from (1.32) by recurrence.

Corollary 1.5. *For any* $u \in D_{\mathcal{L}}$, $U(t) = \int_0^t u(\tau)\,d\tau \in D_{\mathcal{L}}$ *and*

$$\lambda(U) \le \max\{\lambda(u), 0\}, \qquad [\mathcal{L}(U)](s) = [\mathcal{L}(u)](s)/s \qquad \textit{if } \mathrm{Re}(s) > \max\{\lambda(u), 0\}. \tag{1.34}$$

Proof. As u is causal, $\int_0^t u(\tau)\,d\tau = \int_{-\infty}^t u(\tau)\,d\tau = (u * H)(t)$ for any $t > 0$. It then suffices to apply the convolution theorem. □

Theorem 1.6. *(Holomorphy) For any* $u \in D_{\mathcal{L}}$,

$$\textit{the function } \widetilde{u} \textit{ is holomorphic in } \mathbf{C}_{\lambda(u)}, \tag{1.35}$$

$$\forall \lambda > \lambda(u), \textit{the function } \widetilde{u} \textit{ is bounded in the half-plane } \{s \in \mathbf{C} : \mathrm{Re}(s) \ge \lambda\}, \tag{1.36}$$

$$\sup_{\mathrm{Im}(s)\in\mathbf{R}} \widetilde{u}(s) \to 0 \qquad \textit{for } \mathrm{Re}(s) \to +\infty. \tag{1.37}$$

Proof. The proof of holomorphism is analogous to that we saw for the Fourier transform, and is left to the reader. (One may actually differentiate w.r.t. s under the integral.) (1.36) holds true as

$$|\widetilde{u}(s)| \le \int_{\mathbf{R}} e^{-t\lambda} |u(t)|\,dt \qquad \forall \lambda \in \,]\lambda(u), \mathrm{Re}(s)[. \tag{1.38}$$

By passing to the limit as $\mathrm{Re}(s) \to +\infty$ in this inequality via the dominated convergence theorem, (1.37) follows. □

Incidentally, notice that the transformed function $\widetilde{u}$ need not be bounded at the interior of the whole half-plane $\mathbf{C}_{\lambda(u)}$. The transform $\widetilde{H}(s) = 1/s$ is a counterexample.

Remarks. (i) If the Laplace integral absolutely converges for some $s \in \mathbf{C}$, then the same occurs for $s+iy$ for any $y \in \mathbf{R}$. Therefore the set of absolute convergence is either of the form $\{s \in \mathbf{C} :\ \mathrm{Re}(s) > a\}$ or $\{s \in \mathbf{C} :\ \mathrm{Re}(s) \ge a\}$ for some $a \in \,]-\infty, +\infty[$ (or it is the whole $\mathbf{C}$).

(ii) In some cases $\widetilde{u}$ may be extended to a holomorphic function defined on a larger domain than $\mathbf{C}_{\lambda(u)}$. For instance, the transform $\widetilde{H}(s)$ $(= 1/s)$ of the Heaviside function H is only defined for $\mathrm{Re}(s) > \lambda(H) = 0$, although the function $s \mapsto 1/s$ is holomorphic in $\mathbf{C} \setminus \{0\}$.

(iii) The theorem of holomorphism is one of the main differences between the properties of the Laplace and Fourier transforms, as in general Fourier transformed functions need not be holomorphic.

For instance, the function $w : \mathbf{R} \to \mathbf{C} : \xi \mapsto (1+\xi^2)^{-2}$ is an element of L^2, hence it is the Fourier transform of some function $u \in L^2$. However, w has no holomorphic continuation to the whole $\mathbf{C}$, as $(1+s^2)^{-2}$ is not defined for $s = \pm i$.

(iv) In several cases the theorem of holomorphism allows one to perform a computation for real s, and then to extend the result to the convergence half-plane. □

2 Further Properties of the Laplace Transform

For the differentiation of the original function and of its Laplace transform, we have analogous rules to those we saw for the Fourier transform. In one of these formulas the initial value of the function

u occurs. [10] We remind the reader that for any function $v = v(t)$, by Dv we denote the derivative in the sense of distributions (which exists for any v), and by v' the derivative almost everywhere (if existing).

Proposition 2.1. *(Laplace transform and a.e. differentiation) (i) For any $u \in D_{\mathcal{L}}$,*

$$\begin{aligned} & tu(t) \in D_{\mathcal{L}}, \quad \lambda(tu(t)) = \lambda(u), \\ & [\mathcal{L}(u)]'(s) = -[\mathcal{L}(tu(t))](s) \qquad \forall s \in \mathbf{C}_{\lambda(u)}. \end{aligned} \tag{2.1}$$

More generally, for any $n \in \mathbf{N}$,

$$\begin{aligned} & t^n u(t) \in D_{\mathcal{L}}, \quad \lambda(t^n u(t)) = \lambda(u), \\ & [\mathcal{L}(u)]^{(n)}(s) = (-1)^n [\mathcal{L}(t^n u(t))](s) \qquad \forall s \in \mathbf{C}_{\lambda(u)}. \end{aligned} \tag{2.2}$$

(ii) For any $u \in D_{\mathcal{L}}$ that is absolutely continuous in $]0, +\infty[$, if $u' \in D_{\mathcal{L}}$ and if there exists

$$u(0^+) := \lim_{t \to 0^+} u(t) \in \mathbf{C},$$

then

$$[\mathcal{L}(u')](s) = s[\mathcal{L}(u)](s) - u(0^+) \qquad \forall s \in \mathbf{C}_{\lambda(u)} \cap \mathbf{C}_{\lambda(u')}. \tag{2.3}$$

Proof. As we saw, multiplication by t does not modify the convergence abscissa of u. The proof of the formula $\mathcal{L}(u)' = -\mathcal{L}(tu(t))$ (the derivative of a holomorphic function) is straightforward.

Let us next check (2.3). As $e^{-st}u(t) \in L^1_t$ whenever $\mathrm{Re}(s) > \lambda(u)$, there exists a diverging sequence $\{t_n\}$ such that $e^{-st_n}u(t_n) \to 0$ as $n \to +\infty$. By partial integration we then get

$$\begin{aligned} [\mathcal{L}(u')](s) &= \int_0^{+\infty} e^{-st} u'(t)\, dt = \lim_{n\to\infty} \int_0^{t_n} e^{-st} u'(t)\, dt \\ &= \lim_{n\to\infty} \left\{ \int_0^{t_n} s e^{-st} u(t)\, dt + e^{-st_n} u(t_n) \right\} - u(0^+) \\ &= s \int_0^{+\infty} e^{-st} u(t)\, dt - u(0^+) \qquad \forall s \in \mathbf{C}_{\lambda(u)} \cap \mathbf{C}_{\lambda(u')}. \quad \square \end{aligned} \tag{2.4}$$

Remarks. (i) If $u \in D_{\mathcal{L}}$ is absolutely continuous in $]0, +\infty[$, let us denote by u' the causal function that coincides with the ordinary derivative in $]0, +\infty[$. This does not entail $u' \in D_{\mathcal{L}}$. For instance $u_1(t) = t^{-1/2}H(t)$ and $u_2(t) = [\sin \exp(t^2)]H(t)$ are both elements of $D_{\mathcal{L}}$, but $u_i' \notin D_{\mathcal{L}}$ for $i = 1, 2$.

(ii) In part (ii) of the latter theorem, the hypothesis $u' \in D_{\mathcal{L}}$ cannot be dropped. Indeed the a.e. differentiability of u alone does not entail (2.3), as u might have a jump at some $t > 0$. As we know, these discontinuities entail the occurrence of Dirac masses in Du.

The formula (2.3) is easily extended to higher-order derivatives.

Proposition 2.2. *Let $u \in D_{\mathcal{L}}$ be such that $u, u', ..., u^{(m-1)}$ are absolutely continuous in $]0, +\infty[$ for an integer $m > 1$. If $u', ..., u^{(m)} \in D_{\mathcal{L}}$ and if the limits $u(0^+), ..., u^{(m-1)}(0^+)$ exist in $\mathbf{C}$, then*

$$[\mathcal{L}(u^{(m)})](s) = s^m [\mathcal{L}(u)](s) - \sum_{n=0}^{m-1} s^{m-n-1} u^{(n)}(0^+) \qquad \forall s \in \mathbf{C}_{\lambda(u)} \cap ... \cap \mathbf{C}_{\lambda(u^{(m)})}. \tag{2.5}$$

[10] When dealing with the Laplace transform of distributions we shall illustrate this issue.

Proof. For instance, for $m = 2$ by applying Proposition 2.1 first to u' and then to u, we get

$$[\mathcal{L}(u'')](s) = s[\mathcal{L}(u')](s) - u'(0^+) = s^2[\mathcal{L}(u)](s) - su(0^+) - u'(0^+)$$

for any $s \in \mathbf{C}_{\lambda(u)} \cap \mathbf{C}_{\lambda(u')} \cap \mathbf{C}_{\lambda(u'')}$. The equality (2.5) is easily proved by recurrence. □

Proposition 2.3. *(Laplace transform and integration) If $u, u(t)/t \in D_{\mathcal{L}}$ then $\lambda(u(t)/t) = \lambda(u)$ and* [11]

$$[\mathcal{L}(u(t)/t)](s) = \lim_{\mathbf{R}\ni\sigma\to+\infty} \int_s^\sigma [\mathcal{L}(u)](r)\, dr \qquad \forall s \in \mathbf{C}_{\lambda(u)}. \tag{2.6}$$

This limit coincides with the generalized integral of $\mathcal{L}(u)$ between s and $+\infty$ (here $+\infty$ is the limit of $(x, 0)$ as $x \to +\infty$...); this integral does not depend on the integration path, because $\mathcal{L}(u)$ is holomorphic. Anyway it need not converge absolutely, hence it may not be representable as a Lebesgue integral. [12]

Proof. As $\int_0^t u(\tau)d\tau = (u*H)(t)$, the convolution theorem yields

$$\Big[\mathcal{L}\Big(\int_0^t u(\tau)d\tau\Big)\Big](s) = \widetilde{u}(s)\widetilde{H}(s) = \frac{\widetilde{u}(s)}{s} \qquad \text{if } \operatorname{Re}(s) > \max\{\lambda(u), 0\}.$$

On the other hand, if $u, u(t)/t \in D_{\mathcal{L}}$ then by applying the differentiation theorem to $v(t) := u(t)/t$ we have

$$\lambda(u(t)/t) = \lambda(u), \qquad \mathcal{L}(v)' = -\mathcal{L}(tv(t)) = -\mathcal{L}(u),$$

whence by integration

$$\widetilde{v}(s) = \widetilde{v}(\sigma) + \int_s^\sigma [\mathcal{L}(u)](r)\, dr \qquad \forall s \in \mathbf{C}_{\lambda(u)}, \forall \sigma > \lambda(u).$$

Passing to the limit as $\sigma \to +\infty$ and recalling (1.37), we get (2.6). □

The next theorem is an example of the so-called *Tauberian theorems.* These results provide information on the original function u on the basis of the transformed function $\widetilde{u}$, without inverting the Laplace transformation.

Theorem 2.4. **(of final and initial values) For any $u \in D_{\mathcal{L}}$,* [13]

$$\exists u(+\infty) \in \mathbf{C} \quad \Rightarrow \quad \lambda(u) \le 0, \quad s\widetilde{u}(s) \to u(+\infty) \ \text{ as } \ s \to 0, \tag{2.7}$$

$$\exists u(0^+) \in \mathbf{C} \quad \Rightarrow \quad \sup_{\operatorname{Im}(s)\in\mathbf{R}} |s\widetilde{u}(s) - u(0^+)| \to 0 \ \text{ as } \ \operatorname{Re}(s) \to +\infty. \tag{2.8}$$

If $\lambda(u) = 0$ then in (2.7) it is understood that $s \to 0$ from the half-plane of complex numbers with positive real part. The convergence in (2.8) is tantamount to the following: as $\operatorname{Re}(s) \to +\infty$, $s\widetilde{u}(s) \to u(0^+)$ uniformly w.r.t. $\operatorname{Im}(s)$.

*** Proof.** In order to simplify the argument, for both statements we shall assume that

$$u \text{ is absolutely continuous in }]0, +\infty[, \quad u' \in D_{\mathcal{L}}, \quad \exists u(+\infty), \quad \exists u(0^+). \tag{2.9}$$

[11] Here we mean that σ is real; anyway, because of the holomorphism of $\mathcal{L}(u)$, this limit may also be taken as σ varies along a complex path in $\mathbf{C}_{\lambda(u)}$.

[12] In formula (2.6) we do not write $\int_s^{+\infty} [\mathcal{L}(u)](\tau)\, d\tau$, as we reserve the integral notation to the Lebesgue integral.

[13] We set $u(+\infty) := \lim_{t\to+\infty} u(t)$ whenever this limit exists (in $\mathbf{C}$).

As the function u is continuous and converges as $t \to 0$ and $t \to +\infty$, u is bounded and thus $\lambda(u) \le 0$. Because of the dominated convergence theorem, we have

$$\int_0^{+\infty} u'(t)\,dt = \lim_{s\to 0} \int_0^{+\infty} e^{-st} u'(t)\,dt = \lim_{s\to 0}\,[\mathcal{L}(u')](s) \overset{(2.3)}{=} \lim_{s\to 0}\, s\widetilde{u}(s) - u(0^+). \tag{2.10}$$

On the other hand

$$\int_0^{+\infty} u'(\tau)\,d\tau = \lim_{t\to+\infty} \int_0^{t} u'(\tau)\,d\tau = \lim_{t\to+\infty} u(t) - u(0^+) = u(+\infty) - u(0^+), \tag{2.11}$$

and by comparing these two identities we get (2.7).

Similarly, we have

$$\lim_{\mathrm{Re}(s)\to+\infty} s\widetilde{u}(s) - u(0^+) \overset{(2.3)}{=} \lim_{\mathrm{Re}(s)\to+\infty} [\mathcal{L}(u')](s) \overset{(1.37)}{=} 0, \tag{2.12}$$

and this limit is uniform respect to $\mathrm{Im}(s)$. (2.8) is thus proved. □

Remarks. (i) The final-value theorem cannot be inverted: the existence of $\lim_{s\to 0} s\widetilde{u}(s)$ in $\mathbf{C}$ does not entail that of $u(+\infty)$. For instance $u(t) = (\sin t)H(t)$ does not converge as $t \to +\infty$, although $s\widetilde{u}(s) = s/(s^2+1) \to 0$ as $s \to 0$.

(ii) The initial-value theorem also cannot be inverted: the existence of $\lim_{\mathrm{Re}(s)\to+\infty} s\widetilde{u}(s)$ in $\mathbf{C}$ (uniformly w.r.t. $\mathrm{Im}(s)$) does not entail that of $u(0^+)$. We omit the counterexample, which is less simple than the previous one.

Inversion of the Laplace transform. The following theorem provides an explicit formula for the antitransform. This is derived by reducing the Laplace transform to the Fourier transform, see (1.16).

Let us first say that a function $u \in D_{\mathcal{L}}$ is of *exponential order* if

$$\exists \alpha \in \mathbf{R}, \exists M > 0: \qquad |u(t)| \le M e^{\alpha t} \text{ for a.e. } t > 0. \tag{2.13}$$

This class includes most of the functions that occur in applications. Not all elements of $D_{\mathcal{L}}$ are of exponential order, but any nondecreasing $u \in D_{\mathcal{L}}$ has this property. [Ex] For any $u \in D_{\mathcal{L}}$ of exponential order, $\lambda(u)$ is the infimum of the $\alpha \in \mathbf{R}$ that fulfill (2.13).

Theorem 2.5. *(Riemann-Fourier) If $u \in D_{\mathcal{L}}$ is of exponential order, then, denoting by $\widetilde{u}$ its Laplace transform,*

$$\begin{aligned} u(t) &= \frac{1}{2\pi i}\mathrm{P.V.}\int_{x+i\mathbf{R}} e^{st}\widetilde{u}(s)\,ds \\ &\Big(= \frac{1}{2\pi} \lim_{R\to+\infty} \int_{-R}^{R} e^{(x+iy)t}\widetilde{u}(x+iy)\,dy\Big) \qquad q\forall t \in \mathbf{R}, \forall x > \lambda(u). \end{aligned} \tag{2.14}$$

Proof. Let us select any $x > \lambda(u)$ and set

$$\varphi_x(y) := \widetilde{u}(x+iy) = \int_{\mathbf{R}} e^{-(x+iy)t} u(t)\,dt \qquad \forall y \in \mathbf{R},$$

that is, $\varphi_x(y) = \sqrt{2\pi}[\mathcal{F}(e^{-xt}u(t))](y)$. By (2.13), $\lambda(u) \le \alpha$; as $x > \lambda(u)$, then

$$\exists \alpha \in \,]\lambda(u), x[: \qquad M_\alpha := \operatorname*{ess\,sup}_{t>0}\{e^{-\alpha t}|u(t)|\} < +\infty; \qquad [Ex]$$

hence

$$|e^{-(x+iy)t}u(t)| = |e^{-xt}u(t)| = |e^{-\alpha t}u(t)|e^{(\alpha-x)t} \le M_\alpha e^{(\alpha-x)t} \in L^2.$$

The Fourier transform $\mathcal{F}(e^{-xt}u(t))$ may thus be understood not only in the sense of L^1 (i.e., as a Lebesgue integral) but also in the sense of L^2, thus as a principal value. The theorem of inversion of the Fourier transform in L^2 then yields

$$e^{-xt}u(t) = \frac{1}{\sqrt{2\pi}}\big[\mathcal{F}^{-1}(\varphi_x)\big](t) = \frac{1}{2\pi}\text{P.V.}\int_{\mathbf{R}} e^{ity}\varphi_x(y)dy \qquad q\forall t \in \mathbf{R}, \forall x > \lambda(u), \tag{2.15}$$

that is, (2.14). □

An exercise in complex calculus. Next we directly check that the principal value occurring in (2.14) does not depend on $x > \lambda(u)$. Here we assume that

$$\exists\, C, a > 0 : \forall s \in \mathbf{C}_\lambda(u) \qquad |\widetilde{u}(s)| \le C|s|^{-a}, \tag{2.16}$$

although, because of the previous argument, this additional hypothesis is not really needed.

For any x_1, x_2 with $\lambda(u) < x_1 < x_2$, because of Proposition 1.6 the function $s \mapsto e^{st}\widetilde{u}(s)$ is holomorphic in the strip of the complex plane that is comprised between the straight lines $x_1 + i\mathbf{R}$ and $x_2 + i\mathbf{R}$. For any $R > 0$ let us denote by B_R the boundary of the rectangle of vertices $x_1 - iR, x_1 + iR, x_2 + iR, x_2 - iR$. By the classical Cauchy integral theorem,

$$\int_{B_R} e^{st}\widetilde{u}(s)ds = 0 \qquad \forall R > 0. \tag{2.17}$$

Because of (2.16) the modulus of the integral along the horizontal segment $y = R$ is dominated by

$$\int_{x_1}^{x_2} e^{xt}|\widetilde{u}(x+iR)|\,dx \le Ce^{x_2 t}|x_1 + iR|^{-a}(x_2 - x_1),$$

which vanishes as $R \to +\infty$. The same holds for the modulus of the integral along the horizontal segment $y = -R$. (2.17) then yields

$$\int_{x_1-iR}^{x_1+iR} e^{st}\widetilde{u}(s)ds - \int_{x_2-iR}^{x_2+iR} e^{st}\widetilde{u}(s)ds \to 0 \text{ for } R \to +\infty.$$

As both these integrals converge in the sense of the principal value as $R \to +\infty$, hence

$$\frac{1}{2\pi i}\text{P.V.}\int_{x_1+i\mathbf{R}} e^{st}\widetilde{u}(s)\,ds = \frac{1}{2\pi i}\text{P.V.}\int_{x_2+i\mathbf{R}} e^{st}\widetilde{u}(s)\,ds.$$

Remark. The inversion formula requires the evaluation of integrals along paths in the complex field (so-called *contourn integrals),* and this may be cumbersome. In several cases it is more convenient to antitransform a function by expanding it in *simple fractions,* and then use transform tables *backward,* as we shall see in the next section.

This result also entails that the Laplace transform is injective on the whole $D_{\mathcal{L}}$.

Antitransformability. The problem of characterizing the class $\mathcal{L}(D_{\mathcal{L}})$ of transformed functions is nontrivial. Here we just provide a sufficient condition for existence of the inverse transform.

Theorem 2.6. *(Antitransformability-I) The restriction of a function U of complex variable to a suitable half-plane is the Laplace transform of some function $T \in D_{\mathcal{L}}$ if* [14]

$$\exists \lambda \in \mathbf{R}: \; U \text{ is holomorphic for } \operatorname{Re}(s) > \lambda, \tag{2.18}$$

$$\exists \alpha > 1: \; \sup_{y\in\mathbf{R}} |x+iy|^{\alpha} U(x+iy) \to 0 \qquad \text{as } x \to +\infty. \tag{2.19}$$

This implication cannot be inverted. For instance, the condition (2.19) fails for the function $U(s) = 1/s$, which however is the transform of the Heaviside function for $\operatorname{Re}(s) > 0$. [15] This example and the theorem above yield the next result, which is often applied because of the ubiquity of the Heaviside function.

Corollary 2.7. *Let U be a complex function of complex variable which fulfills the hypotheses of Theorem 2.6, then the functions V of the form*

$$V(s) = U(s) + \frac{a}{s} \qquad \forall s \in \mathbf{C}_{\lambda} \text{ (for some } a \in \mathbf{C} \text{ and } \lambda > 0), \tag{2.20}$$

are also antitransformable, and $\mathcal{L}^{-1}(V) = \mathcal{L}^{-1}(U) + aH$.

Therefore any function V that can be represented as a sum of the form (2.20) is antitransformable.

3 Laplace Transform of Distributions

We saw that the Laplace transform extends the Fourier transform in L^1. As the latter transform can be extended to the space $\mathcal{S}'$ of tempered distributions, one may wonder whether the Laplace transform can also be extended to some space of distributions. We shall answer in the affirmative, under suitable restrictions.

Let us define the class $\bar{D}_{\mathcal{L}}$ of transformable distributions with convergence abscissa $\bar{\lambda}$: [16]

$$\bar{D}_{\mathcal{L}} := \big\{ T \in \mathcal{D}' : \operatorname{supp}(T) \subset \mathbf{R}^+, \exists x \in \mathbf{R} : e^{-xt} T(t) \in \mathcal{S}'_t \big\}, \tag{3.1}$$

$$\bar{\lambda}(T) := \inf \big\{ x \in \mathbf{R} : e^{-xt} T(t) \in \mathcal{S}'_t \big\} \; (\in [-\infty, +\infty[) \qquad \forall T \in \bar{D}_{\mathcal{L}}, \tag{3.2}$$

$$\mathbf{C}_{\bar{\lambda}(T)} := \{ s \in \mathbf{C} : \; \operatorname{Re}(s) > \bar{\lambda}(T) \} \qquad \forall T \in \bar{D}_{\mathcal{L}}. \tag{3.3}$$

We are tempted to write a formula like (1.11), simply by replacing the integral with a suitable duality pairing $\langle \cdot, \cdot \rangle$. May we set

$$[\bar{\mathcal{L}}(T)](s) = \langle T(t), e^{-st} \rangle \qquad \forall s \in \mathbf{C}_{\bar{\lambda}(T)}, \forall T \in \bar{D}_{\mathcal{L}} \; ? \tag{3.4}$$

Here we cannot use the duality pairing between $\mathcal{D}'$ and $\mathcal{D}$ as the support of e^{-st} is not compact (and the same holds for $e^{-st}H(s)$). Even if $T \in \mathcal{S}'$, we cannot use the pairing between $\mathcal{S}'$ and $\mathcal{S}$, since e^{-st}

[14] (2.19) is tantamount to $\exists \alpha > 1 : \forall \varepsilon > 0, \exists L > 0$ such that if $\operatorname{Re}(s) > L$ then $|s|^{\alpha} |U(s)| \le \varepsilon$.
This means that, for a suitable $\alpha > 1$, $|s|^{\alpha} U(s) \to 0$ as $\operatorname{Re}(s) \to +\infty$, uniformly w.r.t. $\operatorname{Im}(s)$.

[15] Incidentally, notice that, although the function U is not integrable on $x + i\mathbf{R}$ for $x > 0$, one can apply the Riemann-Fourier formula (2.14), as there the integral is meant in the sense of the principal value.

[16] As we know, the condition $\operatorname{supp}(T) \subset \mathbf{R}^+$ means that $\langle T, v \rangle = 0$ for any $v \in \mathcal{D}$ such that $\operatorname{supp}(v) \subset \,]-\infty, 0[$. We use the notation $\mathcal{S}'_t$ to indicate that t is the independent variable; here x is just a parameter.
At variance of what we did for the Laplace transform of functions, which was defined by a Lebesgue integral, here we do not speak of abscissa of *absolute* convergence, as this is meaningless for duality pairings.

does not decay exponentially as $t \to +\infty$ whenever $\text{Re}(s) \leq 0$. On the other hand, we do not expect to encounter any substantial difficulty for $t \to -\infty$, since the support of T is confined to $\mathbf{R}^+$.

We then use the following construction. First, we fix any function $\zeta \in C^\infty$ such that

$$\zeta = 0 \quad \text{in }]-\infty, a[, \text{ for some } a < 0, \qquad \zeta = 1 \quad \text{in } \mathbf{R}^+.$$

For any $T \in \bar{D}_{\mathcal{L}}$ and any $s \in \mathbf{C}_{\bar{\lambda}(T)}$, we then select any $x \in]\bar{\lambda}(T), \text{Re}(s)[$. Notice that $e^{-xt}T(t) \in \mathcal{S}'$ because of (3.2), and $e^{(x-s)t}\zeta(t) \in \mathcal{S}$ as $\text{Re}(x - s) < 0$. We then set

$$[\bar{\mathcal{L}}(T)](s) := {}_{\mathcal{S}'}\langle e^{-xt}T(t), e^{(x-s)t}\zeta(t)\rangle_{\mathcal{S}} \qquad \forall s \in \mathbf{C}_{\bar{\lambda}(T)}, \forall T \in \bar{D}_{\mathcal{L}}. \tag{3.5}$$

It is not difficult to check that this value does not depend on the choice of $x \in]\bar{\lambda}(T), \text{Re}(s)[$ and of the function ζ. [Ex] This definition is thus *formally* equivalent to (3.4), which may be regarded just as a short-writing of the rigorous formula (3.5).

Here is an important example:

$$[\bar{\mathcal{L}}(\delta_a)](s) = e^{-as} \qquad \forall s \in \mathbf{C}, \forall a \geq 0. \tag{3.6}$$

Notice that $\delta_a := \delta_0(\cdot - a)$ is causal iff $a \geq 0$.

As $L^1 \subset \mathcal{S}'$, the transform $\bar{\mathcal{L}}$ of distributions extends that of functions (that we denoted by $\mathcal{L}$):

$$D_{\mathcal{L}} \subset \bar{D}_{\mathcal{L}}, \tag{3.7}$$

$$\bar{\lambda}(u) \leq \lambda(u) \quad \text{i.e.} \quad \mathbf{C}_{\lambda(u)} \subset \mathbf{C}_{\bar{\lambda}(u)} \qquad \forall u \in D_{\mathcal{L}}, \tag{3.8}$$

$$\bar{\mathcal{L}}(u)\big|_{\mathbf{C}_{\lambda(u)}} = \mathcal{L}(u) \qquad \forall u \in D_{\mathcal{L}}. \tag{3.9}$$

Moreover, the use of duality pairing instead of the (Lebesgue) integral overcomes the restriction of *absolute* convergence.

We shall also see that:

(i) there exists $u \in L^1_{loc}$ such that $u \in \bar{D}_{\mathcal{L}} \setminus D_{\mathcal{L}}$ — thus $\mathcal{L} \neq \bar{\mathcal{L}}\big|_{\bar{D}_{\mathcal{L}} \cap L^1_{loc}}$;

(ii) there exists $u \in D_{\mathcal{L}}$ such that $\bar{\lambda}(u) < \lambda(u)$ — thus $\mathcal{L} \neq \bar{\mathcal{L}}\big|_{D_{\mathcal{L}}}$, despite of (3.9).

Alike what we saw for functions, the Laplace transform of distributions is holomorphic.

Proposition 3.1. *For any $T \in \bar{D}_{\mathcal{L}}$, the function $\bar{\mathcal{L}}(T)$ is holomorphic in $\mathbf{C}_{\bar{\lambda}(T)}$. []*

Anyway (1.36) and (1.37) do not take over to distributions; e.g., $\bar{\mathcal{L}}(D\delta_0) = s$ for any $s \in \mathbf{C}$.

Theorems 1.1, 1.2, 1.3, 1.4 also hold in $\bar{D}_{\mathcal{L}}$. The same applies to the inversion formula, under hypotheses that here we do not display.

Next we extend the transform of the derivative, under weaker hypotheses and by a simpler formula than for functions. We remind the reader that by D we denote the derivative in the sense of distributions.

Theorem 3.2. *(Laplace transform of the distributive derivative) For any $T \in \bar{D}_{\mathcal{L}}$,*

$$DT \in \bar{D}_{\mathcal{L}}, \qquad \bar{\lambda}(DT) \leq \bar{\lambda}(T), \qquad [\bar{\mathcal{L}}(DT)](s) = s[\bar{\mathcal{L}}(T)](s) \quad \forall s \in \mathbf{C}_{\bar{\lambda}(T)}. \tag{3.10}$$

More generally, for any $k \in \mathbf{N}$,

$$D^kT \in \bar{D}_{\mathcal{L}}, \qquad \bar{\lambda}(D^kT) \leq \bar{\lambda}(T), \qquad [\bar{\mathcal{L}}(D^kT)](s) = s^k[\bar{\mathcal{L}}(T)](s) \quad \forall s \in \mathbf{C}_{\bar{\lambda}(T)}. \tag{3.11}$$

Proof. Because of Theorem 1.1, (3.10) stems from the analogous formula for the Fourier transform of distributions. (3.11) then follows by induction. □

For instance,

$$[\bar{\mathcal{L}}(D^k\delta_a)](s) = s^k e^{-as} \qquad \forall s \in \mathbf{C}, \forall a \geq 0, \forall k \in \mathbf{N}. \tag{3.12}$$

Let $u \in D_{\mathcal{L}}$ and assume that u is absolutely continuous in $]0, +\infty[$, that $u' \in D_{\mathcal{L}}$ and that there exists $u(0^+) := \lim_{t\to 0^+} u(t) \in \mathbf{C}$. By part (ii) of Theorem 2.1) then $Du = \bar{\mathcal{L}}(u' + u(0^+)\delta_0)$, and we get

$$s[\bar{\mathcal{L}}(u)](s) \stackrel{(3.10)}{=} \bar{\mathcal{L}}(Du) = \bar{\mathcal{L}}(u' + u(0^+)\delta_0) = \mathcal{L}(u') + u(0^+) \qquad \text{in } \mathbf{C}_{\tilde{\lambda}(u)}. \tag{3.13}$$

We thus retrieve (2.3).

* **Remarks.** (i) There exists $u \in D_{\mathcal{L}}$ such that $\bar{\lambda}(u) < \lambda(u)$. E.g., let us set

$$v(t) = (\sin e^t)H(t) \in D_{\mathcal{L}} \qquad \text{whence} \qquad u(t) := Dv(t) = e^t(\cos e^t)H(t) \in D_{\mathcal{L}}.^{17}$$

It is easy to check that $\lambda(v) = 0$ and $\lambda(u) = 1$. As

$$\bar{\lambda}(u) \stackrel{(3.10)}{\leq} \bar{\lambda}(v) \stackrel{(3.8)}{\leq} \lambda(v) < \lambda(u),$$

we conclude that in this case $\bar{\lambda}(u) < \lambda(u)$. [18]

(ii) There exist locally integrable functions that are elements of $\bar{D}_{\mathcal{L}}$ but not of $D_{\mathcal{L}}$, that is, $\bar{D}_{\mathcal{L}} \cap L^1_{loc} \not\subset D_{\mathcal{L}}$. E.g.,

$$v(t) = (\sin e^{t^2})H(t) \in D_{\mathcal{L}}(\subset \bar{D}_{\mathcal{L}}) \qquad \text{yields} \qquad u(t) = Dv(t) = 2te^{t^2}(\cos e^{t^2})H(t).$$

By Theorem 3.2, $u \in \bar{D}_{\mathcal{L}} \cap L^1_{loc}$ (although at first sight it might not seem so...). But $u \notin D_{\mathcal{L}}$, as the Laplace integral of this function does not converge absolutely. □

Theorem 3.3. *(Antitransformability-II) The restriction of a function U of complex variable to a suitable half-plane is the Laplace transform of some distribution $T \in \bar{D}_{\mathcal{L}}$ iff*

$$\exists \lambda \in \mathbf{R}: \; U \textit{ is holomorphic for } \operatorname{Re}(s) > \lambda, \tag{3.14}$$

$$\exists M > 0, \exists a \geq 0: \; |U(s)| \leq M(1 + |s|)^a \qquad \forall s \in \mathbf{C}_\lambda. \tag{3.15}$$

Notice that this theorem characterizes the transform of distributions, whereas Theorem 2.6 just provides a sufficient condition for a function to be the transform of a function. The reader will notice how far is the condition (2.19) from (3.15); anyway the former entails the latter.

Henceforth we shall drop the bar, and write $\mathcal{L}$, $D_{\mathcal{L}}$, λ, instead of $\bar{\mathcal{L}}$, $\bar{D}_{\mathcal{L}}$, $\bar{\lambda}$. The context will make clear whether we refer to the Laplace transform of functions or of distributions.

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[17] Notice that $u \in \mathcal{S}'$, despite of the occurrence of the exponential, since it is the derivative of a function of $\mathcal{S}'$. Actually u has no genuine exponential growth, because of the oscillating factor.

[18] The abscissa of *absolute* convergence of the a.e. derivative may thus be smaller than that of a transformable function, at variance with what we saw for the Laplace transform of distributions (where of course we deal with derivative in the sense of distributions).

Laplace antitransform of rational functions. Let F be a *rational function* of s, namely the quotient of two polynomials with complex coefficients:

$$F(s) = \frac{P(s)}{Q(s)} = \frac{\sum_{j=0}^{m} a_j s^j}{\sum_{\ell=0}^{n} b_\ell s^\ell} \qquad (a_m, b_n \neq 0,\ m, n \in \mathbf{N}). \tag{3.16}$$

If $m \geq n$ then F can be rewritten in the form

$$F(s) = \sum_{k=0}^{m-n} c_k s^k + \frac{\widetilde{P}(s)}{Q(s)} =: G(s) + R(s) \qquad \text{for } \operatorname{Re}(s) > \lambda; \tag{3.17}$$

here $c_{m-n} = a_m/b_n \neq 0$, $c_0, ..., c_{m-n-1} \in \mathbf{C}$, and $\widetilde{P}(s) = \sum_{j=0}^{\widetilde{m}} \widetilde{a}_j s^j$ is a polynomial of degree $\widetilde{m} < n$. These coefficients may be determined via the identity $P(s) = G(s)Q(s) + \widetilde{P}(s)$, that is,

$$\sum_{j=0}^{m} a_j s^j = \Big(\sum_{k=0}^{m-n} c_k s^k \Big) \Big(\sum_{\ell=0}^{n} b_\ell s^\ell \Big) + \sum_{j=0}^{\widetilde{m}} \widetilde{a}_j s^j \qquad \text{for } \operatorname{Re}(s) > \lambda. \tag{3.18}$$

By eliminating the common factors (if any), we may assume that the polynomials $\widetilde{P}(s)$ and $Q(s)$ are mutually prime.

Here the abscissa λ is just any real number that is larger than the real part of all roots of Q. By Theorem 3.3 it is straightforward to antitransform the function G in the framework of distributions:

$$G(s) = \sum_{k=0}^{m-n} c_k s^k \quad \text{for } \operatorname{Re}(s) > \lambda \quad \Rightarrow \quad \mathcal{L}^{-1}(G) = \sum_{k=0}^{m-n} c_k D^k \delta_0. \tag{3.19}$$

We are left with the calculus of the antitransform of the *proper* rational function $R(s)$, which is also of the form (3.16) but with $\widetilde{m} < n$. Let $\{z_h : h = 1, ..., \ell\}$ be the distinct complex roots of the polynomial $Q(s)$, each of multeplicity r_h; thus $r_1 + ... + r_\ell = n$. As $Q(s) = b_n \Pi_{h=1}^{\ell} (s - z_h)^{r_h}$ for $h = 1, ..., \ell$, the function

$$R(s) = \frac{\widetilde{P}(s)}{b_n \Pi_{h=1}^{\ell} (s - z_h)^{r_h}} \tag{3.20}$$

is defined for any $s \in \mathbf{C}$ different from all z_hs; hence $R(s)$ is defined whenever $\operatorname{Re}(s) > \max\{\operatorname{Re}(z_h) : h = 1, ..., \ell\}$. This function can be decomposed into a sum of *simple fractions:*

$$R(s) = \sum_{h=1}^{\ell} \sum_{k=1}^{r_h} \frac{c_{hk}}{(s - z_h)^k}\,, \tag{3.21}$$

via so-called *partial fraction expansion.* The family of complex coefficients $\{c_{hk}\}$ can be identified by rewriting each addendum of the right member as a sum of fractions, all with the same denominator $\Pi_{h=1}^{\ell} (s - z_h)^{r_h}$.

For instance, let $\frac{1+s+is^2}{(s-1)^3}$ be one of the addenda of $R(s)$. By setting

$$\frac{1 + s + is^2}{(s - 1)^3} = \frac{a}{s - 1} + \frac{b}{(s - 1)^2} + \frac{c}{(s - 1)^3} \tag{3.22}$$

with $a, b, c \in \mathbf{C}$ to be determined, we get

$$\frac{1+s+is^2}{(s-1)^3} = \frac{a(s-1)^2+b(s-1)+c}{(s-1)^3} = \frac{as^2-2as+a+bs-b+c}{(s-1)^3} \tag{3.23}$$

whence $a = i$, $b = 1+2i$, $c = 2+i$.

By (1.21) and (1.26),

$$\mathcal{L}\Big(\frac{t^{k-1}}{(k-1)!}e^{zt}H(t)\Big) = \frac{1}{(s-z)^k} \qquad \forall z \in \mathbf{C}, \forall k \in \mathbf{N} \setminus \{0\}; \tag{3.24}$$

by antitransforming (3.21) one then gets the formula

$$[\mathcal{L}^{-1}(R)](t) = \sum_{h=1}^{\ell}\sum_{k=1}^{r_h} c_{hk}\frac{t^{k-1}}{(k-1)!}e^{z_h t}H(t) \qquad \forall t > 0. \tag{3.25}$$

We have thus proved the following result.

Theorem 3.4. *(of Heaviside) The quotient of any pair of polynomials, $F(s) = \frac{P(s)}{Q(s)}$ (cf. (3.16)), is the Laplace transform of the distribution $T = \mathcal{L}^{-1}(G) + \mathcal{L}^{-1}(R)$, see (3.19) and (3.25). T is a regular distribution (i.e., a function) iff the degree of $P(s)$ is strictly smaller than that of $Q(s)$.*

*** Heaviside formula for simple roots.** Let us assume that all the roots z_h of the polynomial $Q(s)$ are simple, i.e., $Q(s) = b_n \Pi_{n=1}^{n}(s - z_h)$, so that the expansion in simple fractions of R reads

$$R(s) = \frac{\widetilde{P}(s)}{Q(s)} = \sum_{h=1}^{n} \frac{\mu_h}{s - z_h} \,. \tag{3.26}$$

Next we identify the complex coefficients $\mu_1, ..., \mu_n$. For any k, as $Q(z_k) = 0$ we have

$$R(s)(s-z_k) = \frac{\widetilde{P}(s)}{[Q(s)-Q(z_k)]/(s-z_k)} = \mu_k + \sum_{h \neq k} \frac{\mu_h}{s-z_h}(s-z_k) \qquad \text{for } k = 1, ..., n. \tag{3.27}$$

By passing to the limit as $s \to z_k$, we get

$$\lim_{s \to z_k} R(s)(s-z_k) = \frac{\widetilde{P}(z_k)}{Q'(z_k)} = \mu_k \qquad \text{for } k = 1, ..., n. \tag{3.28}$$

The formula (3.26) thus yields the following classical result.

Theorem 3.5. *(Heaviside formula) If the roots z_h of the polynomial $Q(s)$ are all simple, then*

$$[\mathcal{L}^{-1}(\widetilde{P}/Q)](t) = \sum_{h=1}^{n} \frac{\widetilde{P}(z_h)}{Q'(z_h)} e^{z_h t} H(t) \qquad \forall t > 0. \tag{3.29}$$

*** Case of multiple roots.** For instance, let us assume that $Q(s) = (s-z)^r$ for some $z \in \mathbf{C}$ and an integer $r \geq 1$, and let us look for an expansion in simple fractions of the form (3.26). As

$$[D_s^{r-k}(s-z_h)^r R(s)]_{s=z_h} = (r-k)! c_{hk} \qquad \text{for } k = 1, ..., r, h = 1, ..., \ell, \tag{3.30}$$

we can identify the complex coefficients of (3.25):

$$c_{hk} = \frac{1}{(r-k)!}[D_s^{r-k}(s-z_h)^r R(s)]_{s=z_h} \qquad \text{for } k=1,...,r, h=1,...,\ell. \tag{3.31}$$

Exercises

— Check that the function $\mathcal{L}^{-1}(R)$ is bounded if $\mathrm{Re}(z_h) < 0$ for any h, and that if $\mathcal{L}^{-1}(R)$ is bounded then $\mathrm{Re}(z_h) \leq 0$ for any h.

— Check $u * D^n\delta_0 = D^n u$ for any $u \in L^1_{loc}$ and any $n \in \mathbf{N}$.

4 Laplace Transform and Differential Equations

Let us consider the initial-value problem for an ODE, for instance a second-order equation with constant complex coefficients. Let us fix $\alpha, \beta, \gamma, y_0, y_1 \in \mathbf{C}$ ($\alpha \neq 0$), a function $f : \mathbf{R}^+ \to \mathbf{C}$, and let us search for a function $y : \mathbf{R}^+ \to \mathbf{C}$ such that

$$\begin{cases} P(d/dt)y := \alpha y'' + \beta y' + \gamma y = f(t) & \text{for } t > 0, \\ y(0) = y_0, \qquad y'(0) = y_1. \end{cases} \tag{4.1}$$

For $\alpha, \beta, \gamma > 0$, this is the equation of the damped harmonic oscillator, and has a large number applications in physics, engineering and many other disciplines.

Formulation of the Cauchy problem in L^1_{loc}. Let us assume that $f \in D_{\mathcal{L}}$ and search for $y \in D_{\mathcal{L}}$. Assuming that such a solution actually exists and is Laplace-transformable, let us transform both members of the equation, and set $Y := \mathcal{L}(y)$ and $F := \mathcal{L}(f)$. For the moment let us proceed *formally,* postponing the determination of the half-plane of convergence of the transforms. Reminding that

$$\begin{aligned} &\mathcal{L}(y') = s\mathcal{L}(y) - y(0), \\ &\mathcal{L}(y'') = s\mathcal{L}(y') - y'(0) = s^2\mathcal{L}(y) - sy(0) - y'(0), \end{aligned} \tag{4.2}$$

we get the following equation in frequency:

$$\alpha\big[s^2Y(s) - sy_0 - y_1\big] + \beta\big[sY(s) - y_0\big] + \gamma Y(s) = F(s) \qquad \text{for } \mathrm{Re}(s) > \lambda(f). \tag{4.3}$$

Setting

$$P(s) := \alpha s^2 + \beta s + \gamma, \qquad \Phi(s) := \alpha y_0 s + \alpha y_1 + \beta y_0, \tag{4.4}$$

we rewrite the transformed equation (4.3) in the form

$$P(s)Y(s) = F(s) + \Phi(s) \qquad \text{for } \mathrm{Re}(s) > \lambda(f). \tag{4.5}$$

$P(s)$ is called the *characteristic polynomial* of the differential operator $P(d/dt)$; denoting by $s_1, s_2 \in \mathbf{C}$ its (possibly nondistinct) roots, we have

$$P(s) \neq 0 \qquad \text{for } \mathrm{Re}(s) > \max\{\mathrm{Re}(s_1), \mathrm{Re}(s_2)\}.$$

The equation (4.7) is thus equivalent to

$$Y(s) = \frac{F(s)}{P(s)} + \frac{\Phi(s)}{P(s)} \qquad \text{for } \mathrm{Re}(s) > \max\{\mathrm{Re}(s_1), \mathrm{Re}(s_2), \lambda(f)\} =: M, \tag{4.6}$$

and by antitransforming both members we get the solution of the problem (4.1):

$$y = \mathcal{L}^{-1}\Big(\frac{F(s)}{P(s)}\Big) + \mathcal{L}^{-1}\Big(\frac{\Phi(s)}{P(s)}\Big) \qquad \text{in } \mathbf{R}^+. \tag{4.7}$$

The first term depends on the initial data, and represents the *free response* of the system. The second term depends on the source term f, and represents the *forced response.*

By the Convolution Theorem 1.4,

$$\mathcal{L}^{-1}\Big(\frac{F(s)}{P(s)}\Big) = \mathcal{L}^{-1}\Big(\frac{1}{P(s)}\Big) * \mathcal{L}^{-1}(F(s)) = \mathcal{L}^{-1}\Big(\frac{1}{P(s)}\Big) * f. \tag{4.8}$$

and thus

$$y = \mathcal{L}^{-1}\Big(\frac{1}{P(s)}\Big) * f + \mathcal{L}^{-1}\Big(\frac{\Phi(s)}{P(s)}\Big). \qquad \text{in } \mathbf{R}^+. \tag{4.9}$$

The functions $1/P(s)$ and $\Phi(s)/P(s)$ fulfill the assumptions of the Antitransformability Theorem 2.6; hence $y \in D_{\mathcal{L}}$ and $\lambda(y) \leq M$. [19] This justifies the above developments.

Reformulation of the Cauchy problem in $\mathcal{D}'$. First we show that the Cauchy problem (4.1) can be rewritten as a single equation in the sense of distributions.

If $y = y(t)$ is causal and absolutely continuous in $]0,+\infty[$ and if there exists $y(0+) \in \mathbf{C}$, then $y - y(0+)H$ is absolutely continuous in $\mathbf{R}$. Therefore, still denoting by Dy the derivative in the sense of distributions and by dy/dt or y' the almost everywhere derivative,

$$D[y - y(0+)H(t)] = y' \qquad \text{i.e.,} \qquad Dy = y' + y(0+)\delta_0 \qquad \text{in } \mathcal{D}'. \tag{4.10}$$

Analogously, if y' is also absolutely continuous in $]0,+\infty[$ and if there exists $y'(0+) \in \mathbf{C}$, then

$$D[y' - y'(0+)H(t)] = y'' \qquad \text{i.e.,} \qquad D(y') = y'' + y'(0+)\delta_0 \qquad \text{in } \mathcal{D}'. \tag{4.11}$$

Hence

$$D^2y = D(Dy) \overset{(4.10)}{=} D(y') + y(0+)D\delta_0 \overset{(4.11)}{=} y'' + y(0+)D\delta_0 + y'(0+)\delta_0 \qquad \text{in } \mathcal{D}'. \tag{4.12}$$

Defining the operator

$$P(D) := \alpha D^2 + \beta D + \gamma I \qquad \text{in } \mathcal{D}', \tag{4.13}$$

(4.10) and (4.12) yield the formula

$$P(D)y = \alpha y'' + \beta y' + \gamma y + \alpha y(0+)D\delta_0 + [\alpha y'(0+) + \beta y(0+)]\delta_0 \qquad \text{in } \mathcal{D}'. \tag{4.14}$$

This allows one to rewrite the Cauchy problem (4.1) as a single equation on the whole $\mathbf{R}$ in the sense of distributions for the causal function y:

$$P(D)y = f(t) + \alpha y^0 D\delta_0 + [\alpha y^1 + \beta y^0]\delta_0 \qquad \text{in } \mathcal{D}'. \tag{4.15}$$

Remark. A comparison with the solution of ODEs on the whole line via Fourier transform is in order, see [Fourier chapter].

[19] Restrictions on the convergence abscissa have no effect on the properties of the solution.

Use of the Laplace transform in $\mathcal{D}'$. As we just saw, distributions allow one to include the initial data into the forcing term. This allows us to provide a more synthetical approach to the Cauchy problem.

The second member of the equation (4.7) reads

$$G(s) := F(s) + \Phi(s) = F(s) + \alpha y_0 s + \alpha y_1 + \beta y_0 \qquad \text{for } \mathrm{Re}(s) > \lambda(f). \tag{4.16}$$

Notice that by transforming the equation (4.15) the same result is obtained.

The equation (4.7) also reads $P(s)Y(s) = G(s)$, and has the solution

$$Y(s) = \frac{G(s)}{P(s)} \qquad \text{for } \mathrm{Re}(s) > \max\{\mathrm{Re}(s_1), \mathrm{Re}(s_2), \lambda(f)\}. \tag{4.17}$$

Let us apply the antitransform $\mathcal{L}^{-1}$ to this equality. Recalling that

$$\lambda(D^n\delta_0) = -\infty, \qquad \mathcal{L}(D^n\delta_0) = s^n \qquad \forall n \in \mathbf{N}, \tag{4.18}$$

we have

$$\mathcal{L}^{-1}(\alpha y_0 s + \alpha y_1 + \beta y_0) = \alpha y_0 D\delta_0 + (\alpha y_1 + \beta y_0). \tag{4.19}$$

Setting $h = \mathcal{L}^{-1}(1/P(s))$, we thus get the solution of problem (4.1) in the sense of distributions:

$$\begin{aligned} y &= \mathcal{L}^{-1}\Big(\frac{1}{P(s)}(F + \Phi)\Big) = \mathcal{L}^{-1}\Big(\frac{1}{P(s)}\Big) * [\mathcal{L}^{-1}(F) + \mathcal{L}^{-1}(\Phi)] \\ &= h * [f + \alpha y_0 D\delta_0 + (\alpha y_1 + \beta y_0)\delta_0] \\ &= h * f + \alpha y_0 Dh + (\alpha y_1 + \beta y_0)h \qquad \text{in } \mathbf{R}^+. \end{aligned} \tag{4.20}$$

We thus retrieved (4.9), and obtained a more explicit representation of the solution.

The transfer function. The approach of the *linear system theory* can be applied to initial-value problems via the Laplace transform, along a line that is reminiscent of what we saw for differential equations on the whole $\mathbf{R}$ via the Fourier transform. In this case the solution y linearly depends on the initial data y_0, y_1 and on the forcing term f. This is more clear by using the Laplace transform in $\mathcal{D}'$.

Let us first assume that

$$y_0 = y_1 = 0, \tag{4.21}$$

so that the system that maps the *input* f to the *output* u is linear (rather than affine). [20] In the terminology of system theory, the solution operator $L : f \mapsto y$ is called a *continuous filter.* Here we just deal with the filter that is associated to problem (4.1); however this analysis can be extended to higher order problems, and to a much more general set-up — including partial differential equations, integro-differential equations, and so on.

Let $h := L(\delta_0)$ be the *response in time to the unit impulse* δ_0, that is,

$$(P(D)h :=)\ \alpha D^2 h + \beta Dh + \gamma h = \delta_0 \qquad \text{in } \mathcal{D}'(\mathbf{R}), \tag{4.22}$$

or also, because of (4.9),

$$h = \mathcal{L}^{-1}\Big(\frac{1}{P(s)}\Big) * \delta_0 = \mathcal{L}^{-1}\Big(\frac{1}{P(s)}\Big)\ (\in D_{\mathcal{L}}). \tag{4.23}$$

[20] If the initial data are not homogeneous, however via the approach of distributions we can retrieve a *linear* system by including the initial data into the forcing term.

By applying the transform $\mathcal{L}$, we get the *response in frequency to the unit impulse:*

$$[\mathcal{L}(h)](s) = [\mathcal{L}(L(\delta_0))](s) = \frac{1}{P(s)} \qquad \text{in } \mathbf{C}_{L(\delta_0)},$$

The function $[\mathcal{L}(h)](s) = 1/P(s)$ is called the *transfer function* of the linear system L in frequency, or also the *spectrum* of the system L. The antitransform $h = \mathcal{L}^{-1}(1/P(s))$ is thus also called the *transfer function in time* of the system.

Because of (4.22), mathematicians call h the *fundamental solution* of the differential equation $P(D)u = f$. This is nonunique, as it is determined up to the sum of a solution of the homogeneous equation, $P(D)v = 0$. Anyway h is the unique *causal solution,* [21] so that for the Cauchy problem (4.1)

$$\begin{aligned}&\text{the regular function } h \text{ is the unique fundamental solution in } \bar{D}_{\mathcal{L}},\\ &\text{(4.23) is the unique solution in } \bar{D}_{\mathcal{L}} \text{ of the homogeneous Cauchy problem (4.1), (4.21).}\end{aligned} \tag{4.24}$$

Note that this is at variance with what we saw via the Fourier transform for the problem on the whole $\mathbf{R}$, cf. (**??**)[Fourier chapter].

Two examples. Let us fix any $k > 0$, any $f \in D_{\mathcal{L}}$, and consider the differential equations

$$y_1'' - k^2 y_1 = f(t), \qquad y_2'' + k^2 y_2 = f(t); \tag{4.25}$$

the second one represents harmonic motion. These are respectively associated to the operators

$$P_1(D) := D^2 - k^2 I, \qquad P_2(D) := D^2 + k^2 I \qquad (I\text{: identity operator}),$$

with characteristic polynomials

$$P_1(s) := s^2 - k^2, \qquad P_2(s) := s^2 + k^2 \qquad (s \in \mathbf{C}),$$

and with respective roots

$$s_{11}, s_{12} = \pm k, \qquad s_{21}, s_{22} = \pm ik.$$

The general solution y_j $(j = 1, 2)$ of the corresponding problem (4.1) has the form (4.9), with convergence abscissa

$$\lambda(y_1) \le \max\{k, \lambda(f)\}, \qquad \lambda(y_2) \le \max\{0, \lambda(f)\},$$

and transfer function in time

$$\begin{aligned} h_1(s) &= \mathcal{L}^{-1}\Big(\frac{1}{P_1(s)}\Big) = \frac{1}{k}\,\mathcal{L}^{-1}\Big(\frac{k}{s^2 - k^2}\Big) = \frac{\sinh kt}{k} H(t),\\ h_2(s) &= \mathcal{L}^{-1}\Big(\frac{1}{P_2(s)}\Big) = \frac{1}{k}\,\mathcal{L}^{-1}\Big(\frac{k}{s^2 + k^2}\Big) = \frac{\sin kt}{k} H(t). \end{aligned} \tag{4.26}$$

Notice that $h_1 \in \mathcal{D}' \setminus \mathcal{S}'$ and $h_2 \in \mathcal{S}'$.

By (4.20), therefore

$$y_j = h_j * f + \alpha y_j(0) D h_j + \alpha y_j'(0) h_j \qquad \text{in } \mathbf{R}^+, \text{ for } j = 1, 2. \tag{4.27}$$

[21] To prescribe causality is equivalent to prescribing the jump at $t = 0$ of the function and of its derivatives.

*** Higher-order equations.** The previous analysis can be extended to initial-value problems for equations of any order $M \ge 1$.

Let $\alpha_0, ..., \alpha_M, y_0, ..., y_{M-1} \in \mathbf{C}$ ($\alpha_M \ne 0$), $f : \mathbf{R}^+ \to \mathbf{C}$, and let us search for $y : \mathbf{R}^+ \to \mathbf{C}$ such that

$$\begin{cases} \displaystyle\sum_{m=0}^{M} \alpha_m D^m y = f(t) & \text{for } t > 0 \\ D^m y(0) = y_m & \text{for } m = 0, ..., M-1. \end{cases} \tag{4.28}$$

Let us assume that $f, y \in D_{\mathcal{L}}$, set $Y := \mathcal{L}(y)$ and $F := \mathcal{L}(f)$, and apply the Laplace transform to (4.28). Recalling the differentiation Theorem 2.2, (4.28) yields

$$\sum_{m=0}^{M} \alpha_m s^m Y(s) - \sum_{m=0}^{M} \alpha_m \sum_{n=0}^{m-1} s^{m-1-n} y_n = F(s) \qquad \text{for } \operatorname{Re}(s) > \max\{M, \lambda(f)\}. \tag{4.29}$$

(As we saw for the second-order equation, here also we might formulate the Cauchy problem as a single equation in the sense of distributions, getting of course the same transformed equation.) Defining

$$\begin{aligned} P(s) &:= \sum_{m=0}^{M} \alpha_m s^m \qquad (\textit{characteristic polynomial}), \\ \Phi(s) &:= \sum_{m=0}^{M} \alpha_m \sum_{n=0}^{m-1} s^{m-1-n} y_n, \end{aligned} \tag{4.30}$$

the equation (4.29) reads

$$P(s) Y(s) = F(s) + \Phi(s) \qquad \text{for } \operatorname{Re}(s) > \lambda(f). \tag{4.31}$$

Denoting by $s_1, ..., s_N \in \mathbf{C}$ the (possibly nondistinct) complex roots of $P(s)$,

$$P(s) \ne 0 \qquad \text{for } \operatorname{Re}(s) > M := \max\{\operatorname{Re}(s_i) : i = 1, ..., N\}.$$

The equation (4.29) is thus equivalent to

$$Y(s) = \frac{F(s)}{P(s)} + \frac{\Phi(s)}{P(s)} \qquad \text{for } \operatorname{Re}(s) > \max\{M, \lambda(f)\}.$$

Defining the transfer function in time $h := \mathcal{L}^{-1}(1/P(s))$, the Convolution Theorem 1.4 and $(4.30)_2$ yield

$$\begin{aligned} y &= \mathcal{L}^{-1}\Big(\frac{1}{P(s)} F\Big) + \mathcal{L}^{-1}\Big(\frac{1}{P(s)} \Phi\Big) = h * \mathcal{L}^{-1}(F) + h * \mathcal{L}^{-1}(\Phi) \\ &= h * f + \sum_{m=0}^{M} \alpha_m \sum_{n=0}^{m-1} y_n \, h * D^{m-1-n} \delta_0 \\ &= h * f + \sum_{m=0}^{M} \alpha_m \sum_{n=0}^{m-1} y_n \, D^{m-1-n} h \qquad \text{in } \mathbf{R}^+. \end{aligned} \tag{4.32}$$

This is the sum of the free and forced responses. The function y has convergence abscissa $\lambda(y) \le \max\{M, \lambda(f)\}$. This justifies the previous transformations.

This discussion can be extended in several directions: e.g., the Cauchy problem for systems of differential equations, equations and systems of differential equations with delay, and so on.

*** Linear systems of first-order equations.** Let

$$A \in \mathbf{C}^{N\times N} \ \ (N \geq 1), \quad f : \mathbf{R}^+ \to \mathbf{C}^N, \quad f \in L^1_{loc}, \quad u^0 \in \mathbf{C}^N,$$

and consider the following first-order vectorial Cauchy problem, for the unknown function $u : \mathbf{R}^+ \to \mathbf{C}^N$:

$$\begin{cases} D_t u = A \cdot u + f & \text{in } \mathbf{R}^+, \\ u(0) = u^0. \end{cases} \tag{4.33}$$

By extending u and f to causal (vectorial) functions, this problem can be formulated as the single equation

$$D_t u = A \cdot u + f + u^0 \delta_0 \qquad \text{in } \mathcal{D}'(\mathbf{R})^N. \tag{4.34}$$

Let us assume that f and u are both Laplace-transformable (componentwise), and set $U = \mathcal{L}(u)$, $F = \mathcal{L}(f)$. By transforming the equation (4.34), we get

$$(sI - A) \cdot U(s) = F(s) + u^0 \qquad \text{for } \operatorname{Re}(s) > \max_{j=1,\ldots,N} \{\lambda(u_j), \lambda(f_j)\}, \tag{4.35}$$

that is, defining the *resolvent matrix* $R(s) = (sI - A)^{-1}$,

$$U(s) = R(s) \cdot [F(s) + u^0] \qquad \text{for } \operatorname{Re}(s) \text{ large enough.} \tag{4.36}$$

More precisely, defining the (possibly nondistinct) complex eigenvalues $s_1, ..., s_N$ of the matrix A, (4.36) holds for $\operatorname{Re}(s) > \max_{j=1,\ldots,N} \{\lambda(u_j), \lambda(f_j), \operatorname{Re}(s_j)\}$.

Let us now introduce the (causal) matrix antitransform $S = \mathcal{L}^{-1}(R) : \mathbf{R} \to \mathbf{C}^{N\times N}$, namely, the solution of the problem

$$(sI - A)^{-1} = [\mathcal{L}(S)](s) \qquad \text{for } \operatorname{Re}(s) > \max_{j=1,\ldots,N} \{\operatorname{Re}(s_j)\}. \tag{4.37}$$

Antitransforming the equation (4.36) we get $u(t) = \mathcal{L}^{-1}(R) * \mathcal{L}^{-1}(F + u^0)$. We thus retrieve the classical *formula of variation of parameters:*

$$u(t) = S(t) * f(t) + S(t) * (u^0 \delta_0) = \int_0^t S(t - \tau) \cdot f(\tau)\, d\tau + S(t) \cdot u^0 \qquad \forall t \in \mathbf{R}. \tag{4.38}$$

The function S represents the *semigroup* that is determined by the matrix A. [22] The equations (4.37) and (4.38) can then be interpreted as

the resolvent $(sI - A)^{-1}$ of the semigroup $S(t)$
coincides with the Laplace transform of that semigroup. (4.39)

Notice that, using the definition of exponential of a matrix,

$$S(t) = e^{At} H(t) = \sum_{n=0}^{\infty} \frac{A^n t^n}{n!} H(t) \ (\in \mathbf{C}^{N\times N}) \qquad \forall t \in \mathbf{R}. \tag{4.40}$$

[22] A semigroup is a family of operators $S : [0, +\infty[\to \mathcal{L}(X)$ ($\mathcal{L}(X)$ being the space of linear and continuous operators on a Banach space X) such that $S(t_1 + t_2) = S(t_1) \circ S(t_2)$ for any $t_1, t_2 \geq 0$.

*** Linear systems of higher-order equations.** Let A, f be as above, $m \in \mathbf{N}$, $a_n, u^n \in \mathbf{C}$ for $n = 0, ..., m-1$ ($a_m \neq 0$), and consider the following vector Cauchy problem of order m:

$$\begin{cases} \sum_{n=0}^m a_n D^n u(t) = A \cdot u(t) + f(t) \qquad \text{for } t > 0, \\ D^n u(0) = u_n \qquad \text{for } n = 0, ..., m-1. \end{cases} \tag{4.41}$$

By extending u and f to causal vector functions, this problem can be formulated as the single vector equation

$$\sum_{n=0}^{m} a_n D^n u(t) = A \cdot u(t) + f(t) + \sum_{n=0}^{m-1} a_n u_n D^{m-1-n} \delta_0 \qquad \text{in } \mathcal{D}'(\mathbf{R})^N. \tag{4.42}$$

Let us assume that $f, y \in D_{\mathcal{L}}$, set $U = \mathcal{L}(u)$ and $F = \mathcal{L}(f)$; let us also define the characteristic polynomial $P(s)$ and the polynomial $\Phi(s)$ as follows:

$$P(s) = \sum_{n=0}^{m} a_n s^n, \qquad \Phi(s) = \sum_{n=1}^{m} \sum_{j=0}^{n-1} a_n u_j s^{n-h-1} \quad \forall s \in \mathbf{C}. \tag{4.43}$$

By transforming the equation (4.43), we get

$$[P(s)I - A] \cdot U(s) = F(s) + \Phi(s) \qquad \text{for } \operatorname{Re}(s) > \max\{\lambda(u), \lambda(f)\}, \tag{4.44}$$

that is, defining the matrix function $R(s) = [P(s)I - A]^{-1}$,

$$U(s) = R(s) \cdot [F(s) + \Phi(s)] \qquad \text{for } \operatorname{Re}(s) \text{ large enough.} \tag{4.45}$$

Let us introduce the matrix antitransform $S = \mathcal{L}^{-1}(R) : \mathbf{R} \to \mathbf{C}^{N \times N}$, that is, the solution of the problem

$$[P(s)I - A]^{-1} = [\mathcal{L}(S)](s) \qquad \text{for } \operatorname{Re}(s) \text{ large enough.} \tag{4.46}$$

Antitransforming the (4.45), we get

$$u = S * f + S * \mathcal{L}^{-1}(\Phi) = S * f + S * \sum_{n=1}^{m} \sum_{j=0}^{n-1} a_n u_j D^{n-h-1} h \qquad \text{in } \mathbf{R}. \tag{4.47}$$

This approach can be extended to linear evolutionary PDEs, e.g. of the form $D_t u + Au = f$; in this case the associated stationary operator A, in an infinite-dimensional space (typically a Sobolev space).