Laplace transform

Nota: Una trattazione elementare ma più ampia della trasformazione di Laplace è offerta ad esempio dal primo capitolo di

M. Marini: *Metodi matematici per lo studio delle reti elettriche.* C.E.D.A.M., Padova, 1999.

La seguente trattazione è basata in parte sul testo

G. Gilardi: *Analisi tre.* McGraw-Hill, Milano 1994.

Il seguente testo può pure essere utile

G. Folland: *Fourier Analysis and Its Applications.*

This chapter includes the following sections:

1. Laplace transform of functions
2. Further Properties of the Laplace Transform
3. Laplace Transform of Distributions
4. Laplace Antitransform of Rational Functions
5. Laplace Transform and Differential Equations
6. Harmonic Oscillator

1 Laplace Transform of Functions

This transform is strictly related to the Fourier transform, and like the latter it allows one to transform ODEs to algebraic equations. But the Laplace transform is especially suited for initial-value problems, whereas the Fourier transform is appropriate for problems on the whole real line. In applications the theory of Laplace transform also accounts for what is called *symbolic* or *operational calculus,* that was pioneered by O. Heaviside at the end of the 19th century.

From Fourier to Laplace transform. Let us first present some informal remarks for $N = 1$, denoting the independent variable by t. Let us denote by $\widehat{u}$ the Fourier transform of a transformable function $u : \mathbb{R} \to \mathbb{C}$:

$$\widehat{u}(\xi) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} u(t) e^{-i\xi t}\, dt \qquad \text{for } \ \xi \in \mathbb{R}. \tag{1.1}$$

The inversion formula is one of the main elements of interest of this transform:

$$u(t) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \widehat{u}(\xi) e^{i\xi t}\, d\xi \qquad \text{for } \ t \in \mathbb{R}. \tag{1.2}$$

This represents u as an (integral) average of the periodic functions $w_\xi : \mathbb{R} \to \mathbb{C} : t \mapsto e^{i\xi t}$ parameterized by $\xi \in \mathbb{R}$ and with weight $\widehat{u}(\xi)$. Each of the functions $w = w_\xi$ is an eigenfunction of the harmonic operator $-D^2$ in L^∞, and is then called a *harmonic:*

$$- w''(t) = \xi^2 w(t) \qquad \text{for } \ t \in \mathbb{R}. \tag{1.3}$$

Because of the classical method of *variation of the constants,* the solutions of this equation can be used for the study of the nonhomogeneous equation $w''(t)+\xi^2 w(t) = f(t)$ for $t \in \mathbb{R}$, for a prescribed function f. (More specifically, one may look for a solution of the form $w = zw_\xi$, with w_ξ solution of (1.3) and $z = z(t)$ to be determined. By replacing this expression in (1.3), one gets the new equation $z''w + 2z'w' = f$, from which one can determine the function z.)

As $|w_\xi(t)| \leq 1$ for any $\xi, t \in \mathbb{R}$, the Fourier integral (1.2) converges for any $u \in L^1$. Anyway we have seen that one can also assume $u \in L^2$, provided that the Fourier integral is understood in the sense of the principal value of Cauchy; and the domain of this transform can be further extended.

For any $\eta \in \mathbb{R}$, let us also consider the eigenfunctions of the first derivative:

$$w'(t) = \eta w(t) \qquad \text{for } \ t \in \mathbb{R}. \tag{1.4}$$

The solutions of this equation are proportional to $w_\eta : \mathbb{R} \to \mathbb{R} : t \mapsto e^{-\eta t}$; if $\eta \neq 0$, this function diverges exponentially as $t \to +\infty$ or $t \to -\infty$, depending on the sign of η. Analogously to (1.3), one can construct a solution of the nonhomogeneous equations $w'(t) - \eta w(t) = f(t)$ via the classical method of variation of the constants.

In analogy with (1.1)), for the study of this equation first we set

$$\widetilde{u}(\eta) := \int_{\mathbb{R}} u(t) e^{\eta t}\, dt \qquad \text{for } \ \eta \in \mathbb{R}. \tag{1.5}$$

For any $\eta > 0$ ($\eta < 0$, respect.) this integral converges only if $|u(t)|$ exponentially decays to 0 as $t \to -\infty$ (as $t \to +\infty$, respect.). Anyway in many problems of applicative interest the equation (1.4) is not set on the whole $\mathbb{R}$, but just on $\mathbb{R}^+$; in this case it is convenient to restrict oneself to functions that vanish for any $t < 0$. In *Signal Analysis* these are called *causal signals.*

More generally, we can replace the real variable η by a complex variable s. We can then consider the kernels $\mathbb{R} \to \mathbb{C} : t \mapsto e^{-st}$, parameterized by $s \in \mathbb{C}$. For imaginary s we thus retrieve periodic functions, whereas for real s these functions either grow or decay exponentially. Let us then define the Laplace transform

$$u \mapsto \widetilde{u}(s) := \int_{\mathbb{R}} u(t) e^{-st}\, dt \qquad \text{for } \ s \in \mathbb{C}, \tag{1.6}$$

which includes (1.5) for s real and the Fourier transform for s imaginary (the conventional factor $1/\sqrt{2\pi}$ apart). Concerning the convergence of this integral, what we said for (1.5) holds here also, and we shall confine ourselves to causal functions.

Setting $s = x + iy$, we thus have

$$U_x(y) := \widetilde{u}(x + iy) = \int_{\mathbb{R}} u(t) e^{-xt} e^{-iyt}\, dt \qquad \text{for } \ x, y \in \mathbb{R}. \tag{1.7}$$

The variable s is often referred to as a (complex) frequency, although just its imaginary part has the physical meaning of frequency, cf. (1.2). As U_x is the Fourier transform of the function $t \mapsto \sqrt{2\pi}\, u(t) e^{-xt}$, (1.6) may be regarded as the Fourier transform of an array of inputs parameterized by $x \in \mathbb{R}$.

The right side of (1.7) has a meaning whenever $e^{-xt}u(t) \in L^1$; if u is causal, the larger is x the less restrictive is this condition. This allows one to apply this formulation to the Laplace transform also to nonintegrable causal functions.

Functional set-up. The Laplace transform deals with functions of a real variable t, Let us define the class $D_{\mathcal{L}}$ of the *transformable functions,* and for any $u \in D_{\mathcal{L}}$ the *abscissa of (absolute)* *convergence* $\lambda(u)$, the *convergence half-plane* $\mathbb{C}_{\lambda(u)}$, and finally the *Laplace transform* $\mathcal{L}(u)$: [1]

$$D_{\mathcal{L}} := \big\{u \in L^1_{\rm loc} : u(t) = 0 \text{ for a.e. } t < 0, \exists x \in \mathbb{R} : e^{-xt}u(t) \in L^1_t\big\}, \tag{1.8}$$

$$\lambda(u) := \inf \big\{x \in \mathbb{R} : e^{-xt}u(t) \in L^1_t\big\} \ (\in [-\infty, +\infty[) \qquad \forall u \in D_{\mathcal{L}}, \tag{1.9}$$

$$\mathbb{C}_{\lambda(u)} := \{s \in \mathbb{C} : \ \mathrm{Re}(s) > \lambda(u)\} \qquad \forall u \in D_{\mathcal{L}}, \tag{1.10}$$

$$[\mathcal{L}(u)](s) := \int_{\mathbb{R}} e^{-st}u(t)\, dt \qquad \forall s \in \mathbb{C}_{\lambda(u)}, \forall u \in D_{\mathcal{L}}. \tag{1.11}$$

$D_{\mathcal{L}}$ is the set of locally integrable causal functions that have at most exponential growth. The transformed function $\mathcal{L}(u)$ is also called the *Laplace integral.*

Remarks 1.1 (i) Here we speak of abscissa of *absolute* convergence, since any Lebesgue integral is absolutely convergent. Some authors define the Laplace transform as an improper Riemann-type integral, rather than as a Lebesgue integral, and accordingly do not prescribe *absolute* convergence. This has some consequences on some properties of the transform, and has some analogy with what we shall see extending the Laplace transform to distributions.

(ii) What we defined is also called the *unilateral* Laplace transform, as the domain of convergence is a half-plane. In literature a *bilateral* Laplace transform is also defined: in this case u need not be causal. The domain of convergence is then a strip of the form $\{s \in \mathbb{C} : \lambda_1(u) < \mathrm{Re}(s) < \lambda_2(u)\}$, with $-\infty \le \lambda_1(u) < \lambda_2(u) \le +\infty$. This bilateral transform is much less used than the unilateral one, and here we shall not deal with it.

(iii) We do not exclude $\mathbb{C}_{\lambda(u)} = \mathbb{C}$, which we still call a convergence *half-plane.* Indeed $\lambda(u) = -\infty$ if the transformable functions decays more than exponentially; in particular this occurs for any compactly supported function of $D_{\mathcal{L}}$. On the other hand, we exclude $\mathbb{C}_{\lambda(u)} = \emptyset$, i.e. $\lambda(u) = +\infty$.

(iv) The Lebesgue integral (1.11) converges for any $s \in \mathbb{C}_{\lambda(u)}$. Although for some functions this integral may converge also for some complex s with $\mathrm{Re}(s) \le \lambda(u)$, we define the Laplace transformed function $\mathcal{L}(u)$ just in $\mathbb{C}_{\lambda(u)}$, since some properties might fail if $\mathrm{Re}(s) \le \lambda(u)$. For instance, the function $1/s$ is definite for any complex $s \neq 0$, but (as we shall see) $\widetilde{H}(s) = 1/s$ only if $\mathrm{Re}(s) > \lambda(H) = 0$.

* (v) The linear space $D_{\mathcal{L}}$ can be equipped with the following notion of sequential convergence: a sequence $\{u_n\}$ of $D_{\mathcal{L}}$ is said to converge to some $u \in D_{\mathcal{L}}$ in this space iff

(1) $\lambda(u) = \liminf_{n\to+\infty} \ \lambda(u_n) > -\infty$,

(2) $e^{-xt}u_n(t) \to e^{-xt}u(t)$ in L^1_t for any $x > \lambda(u)$.

It is easy to see that this entails $\mathcal{L}(u_n) \to \mathcal{L}(u)$ pointwise in $\mathbb{C}_{\lambda(u)}$, for any $u \in D_{\mathcal{L}}$. □

$D_{\mathcal{L}}$ is a linear space, and the Laplace transform is obviously linear: for any $u, v \in D_{\mathcal{L}}$ ed ogni $\mu_1, \mu_2 \in \mathbb{C}$,

$$\lambda(\mu_1 u + \mu_2 v) \le \max\{\lambda(u), \lambda(v)\}, \quad \mathcal{L}(\mu_1 u + \mu_2 v) = \mu_1\mathcal{L}(u) + \mu_2\mathcal{L}(v). \tag{1.12}$$

In the definition of the elements of $D_{\mathcal{L}}$ we shall often encounter the *Heaviside function* H (also called *unit step):*

$$H(t) := 0 \qquad \forall t \le 0, \qquad H(t) := 1 \qquad \forall t > 0.$$

[1] We write L^1_t in order to make clear that t is the integration variable.

Although in literature the definitions of Fourier coefficients and of Fourier transform may differ in the choice of constants, for the definition of the Laplace transform it seems that there is some agreement.

The occurrence of the factor $H(t)$ will guarantee causality. It is easily checked that

$$e^{-xt}H(t) \in L^1_t \quad \Leftrightarrow \quad \mathrm{Re}(\alpha) > -1 \qquad (\alpha \in \mathbb{C}, x \in \mathbb{R})$$

(note that $t^a H(t) \notin L^1_{\mathrm{loc}}$ if $\mathrm{Re}(\alpha) \le -1$). Therefore

$$\begin{aligned} &H(t) \in D_{\mathcal{L}}, \quad \lambda(H) = 0, \\ &t^\alpha H(t) \in D_{\mathcal{L}}, \quad \lambda(t^\alpha H(t)) = 0 \quad \Leftrightarrow \quad \mathrm{Re}(\alpha) > -1; \end{aligned} \tag{1.13}$$

$$e^{-t^2}H(t) \in D_{\mathcal{L}}, \quad \lambda(e^{-t^2}H(t)) = -\infty; \qquad e^{t^2}H(t), t^{-1}H(t) \notin D_{\mathcal{L}}. \tag{1.14}$$

Relation between the Fourier and Laplace transforms. (Here we define $\mathcal{F}$ as we did in Section 3 of the Fourier chapter.)

• **Theorem 1.2** *For any* $u \in D_{\mathcal{L}}$,

$$[\mathcal{L}(u)](x+iy) = \sqrt{2\pi}\big[\mathcal{F}(e^{-xt}u(t))\big](y) \qquad \forall y \in \mathbb{R}, \forall x > \lambda(u). \tag{1.15}$$

Vice versa, $u \in D_{\mathcal{L}}$ *and* $\lambda(u) \le 0$ *for any causal* $u \in L^1$. *If* $u \in L^1$ *and* $\lambda(u) < 0$ *then* [2]

$$[\mathcal{F}(u)](y) = \frac{1}{\sqrt{2\pi}}[\mathcal{L}(u)](iy) \qquad \forall y \in \mathbb{R}. \tag{1.16}$$

By the foregoing result, several properties of the Fourier transform are easily extended to the Laplace transform. This also applies to the next statement.

Proposition 1.3 *For any* $u \in D_{\mathcal{L}}$,

$$v(t) = u(t-t_0) \quad \Rightarrow \quad \lambda(v) = \lambda(u), \quad \widetilde{v}(s) = e^{-t_0 s}\,\widetilde{u}(s) \qquad \forall t_0 > 0, \tag{1.17}$$

$$v(t) = e^{s_0 t}u(t) \quad \Rightarrow \quad \lambda(v) = \lambda(u) + \mathrm{Re}(s_0), \quad \widetilde{v}(s) = \widetilde{u}(s - s_0) \qquad \forall s_0 \in \mathbb{C}, \tag{1.18}$$

$$v(t) = u(\omega t) \quad \Rightarrow \quad \lambda(v) = \omega\lambda(u), \quad \widetilde{v}(s) = \tfrac{1}{\omega}\widetilde{u}\Big(\tfrac{s}{\omega}\Big) \qquad \forall \omega > 0. \tag{1.19}$$

The proof is left to the Reader. The assertions about convergence abscissas are easily checked: the delay does not modify the behaviour of the function as $t \to +\infty$, whereas the presence of the exponential factor $e^{s_0 t}$ entails a translation of the convergence abscissa.

Examples. For any $u \in D_{\mathcal{L}}$,

$$u(t) = H(t) \quad \Rightarrow \quad \lambda(u) = 0, \quad \widetilde{u}(s) = \frac{1}{s}\ , \tag{1.20}$$

$$u(t) = e^{\gamma t}H(t)\ (\gamma \in \mathbb{C}) \quad \Rightarrow \quad \lambda(u) = \mathrm{Re}(\gamma), \quad \widetilde{u}(s) = \frac{1}{s-\gamma}\ , \tag{1.21}$$

$$u(t) = \cos(\omega t)H(t)\ (\omega \in \mathbb{R}) \quad \Rightarrow \quad \lambda(u) = 0, \quad \widetilde{u}(s) = \frac{s}{s^2+\omega^2}\ , \tag{1.22}$$

$$u(t) = \sin(\omega t)H(t)\ (\omega \in \mathbb{R}) \quad \Rightarrow \quad \lambda(u) = 0, \quad \widetilde{u}(s) = \frac{w}{s^2+\omega^2}\ , \tag{1.23}$$

$$u(t) = \cosh(\omega t)H(t)\ (\omega \in \mathbb{R}) \quad \Rightarrow \quad \lambda(u) = |\omega|, \quad \widetilde{u}(s) = \frac{s}{s^2-\omega^2}\ , \tag{1.24}$$

$$u(t) = \sinh(\omega t)H(t)\ (\omega \in \mathbb{R}) \quad \Rightarrow \quad \lambda(u) = |\omega|, \quad \widetilde{u}(s) = \frac{w}{s^2-\omega^2}\ , \tag{1.25}$$

$$u(t) = t^k H(t)\ (k \in \mathbb{N}) \quad \Rightarrow \quad \lambda(u) = 0, \quad \widetilde{u}(s) = \frac{k!}{s^{k+1}}\ . \tag{1.26}$$

[2] If $u \in L^1$ and $\lambda(u) = 0$ there is a formal inconvenience: $[\mathcal{F}(u)](y)$ exists for a.e. $y \in \mathbb{R}$, but it is not legitimate to write $[\mathcal{L}(u)](iy)$, since $\mathcal{L}(u)$ is defined only in the open half-plane $\mathbb{C}_{\lambda(u)} = \mathbb{C}_0$.

The Reader is invited to check these properties. For instance,

$$\mathcal{L}(\cos(\omega t)H(t)) = \frac{1}{2}\mathcal{L}(e^{i\omega t}H(t) + e^{-i\omega t}H(t)) \overset{(1.21)}{=} \frac{1}{2}\Big(\frac{1}{s-i\omega} + \frac{1}{s+i\omega}\Big) = \frac{s}{s^2+\omega^2}. \qquad (1.27)$$

Hence

$$\mathcal{L}(\cosh(\omega t)H(t)) = \mathcal{L}(\cos(i\omega t)H(t)) = \frac{s}{s^2-\omega^2}. \qquad (1.28)$$

Remark 1.4 (1.26) is generalized by

$$u(t) = t^a H(t)\ (a > -1) \quad \Rightarrow \quad \lambda(u) = 0, \quad \widetilde{u}(s) = \frac{\Gamma(a+1)}{s^{a+1}}, \qquad (1.29)$$

since, by definition of the classical Euler function Γ,

$$\widetilde{u}(s) = \int_0^{+\infty} e^{-st} t^a \, dt = \frac{1}{s^{a+1}} \int_0^{+\infty} e^{-y} y^a \, dy =: \frac{\Gamma(a+1)}{s^{a+1}} \qquad \forall s \in \mathbb{C}_{\lambda(u)} = \mathbb{C}_0. \qquad \square \qquad (1.30)$$

Theorem 1.5 *(Periodic Functions) Let $T > 0$ and $u \in D_{\mathcal{L}}$, $u \not\equiv 0$ be such that $u(t+T) = u(t)$ for any $t > 0$. Then $\lambda(u) = 0$ and, setting $w := u\chi_{]0,T[}$,*

$$\widetilde{u}(s) = \frac{1}{1-e^{-sT}} \widetilde{w}(s) \qquad \forall s \in \mathbb{C}_0. \qquad (1.31)$$

Note that $w \in L^1(0,T)$ and $1 - e^{-sT} \neq 0$ for any $s \in \mathbb{C}_0$.

Proof. Let us set $C := \int_0^T |w(\tau)| \, d\tau$ ($\neq 0$ as $u \neq 0$). As u is periodic and integrable in any interval of the form $]nT, (n+1)T[$, $e^{-xt}u(t) \in L^1$ for all $x > 0$ and for no $x < 0$. Therefore $\lambda(u) = 0$.

Setting $u_T(t) := u(t-T)$ for any $t \in \mathbb{R}$ and noticing that $w = u - u_T$, we have

$$\widetilde{w}(s) = \widetilde{u}(s) - \widetilde{u_T}(s) \overset{(1.17)}{=} \widetilde{u}(s) - e^{-sT}\widetilde{u}(s) = (1-e^{-sT})\widetilde{u}(s) \qquad \forall s \in \mathbb{C}_{\lambda(u)},$$

that is (1.31). $\square$

• **Theorem 1.6** *(Convolution) For any $u, v \in D_{\mathcal{L}}$, $u * v \in D_{\mathcal{L}}$ and*

$$\lambda(u*v) \leq \max\{\lambda(u), \lambda(v)\}, \qquad \mathcal{L}(u*v) = \mathcal{L}(u)\, \mathcal{L}(v). \qquad (1.32)$$

More generally, for any integer $N \geq 2$ and for any $u_1, ..., u_N \in D_{\mathcal{L}}$, we have $u_1 * ... * u_N \in D_{\mathcal{L}}$ and

$$\lambda(u_1 * ... * u_N) \leq \max\{\lambda(u_i) : i = 1, ..., N\}, \qquad \mathcal{L}(u_1 * ... * u_N) = \prod_{i=1}^N \mathcal{L}(u_i). \qquad (1.33)$$

Proof. For any $x > \lambda(u)$ and any $t \in \mathbb{R}$, let us set $\widetilde{u}(t) = e^{-xt}u(t)$ and $\widetilde{v}(t) = e^{-xt}v(t)$. By (1.15), (1.32)follows from applying the analogous property of the Fourier transform (see Theorem **??**) to these functions. (1.33) follows from (1.32) by recurrence. $\square$

The inequality (1.32) may be strict: e.g. consider $u \equiv 0$ and any $v \in D_{\mathcal{L}}$.

Corollary 1.7 *For any $u \in D_{\mathcal{L}}$, $U(t) := \int_0^t u(\tau)\, d\tau \in D_{\mathcal{L}}$ and*

$$\lambda(U) \leq \max\{\lambda(u), 0\}, \qquad [\mathcal{L}(U)](s) = [\mathcal{L}(u)](s)/s \qquad \textit{if } \operatorname{Re}(s) > \max\{\lambda(u), 0\}. \qquad (1.34)$$

Proof. As u is causal, $\int_0^t u(\tau)\,d\tau = (u*H)(t)$ for any $t > 0$. It then suffices to apply the convolution theorem. □

• **Theorem 1.8** *(Holomorphism) For any $u \in D_{\mathcal{L}}$, the Laplace transform $\widetilde{u}$ is holomorphic in $\mathbb{C}_{\lambda(u)}$.*

Proof. It suffices to notice that, in a neighbourhood of any $\widetilde{s} \in \mathbb{C}_{\lambda(u)}$, $\widetilde{u}$ coincides with the Fourier-Laplace transform. As the integrand decays exponentially, by the argument of Theorem **??** [Fourier chapter] we infer that $\widetilde{u}$ is holomorphic. □

Proposition 1.9 *For any $u \in D_{\mathcal{L}}$,*

$$\forall \mu > \lambda(u), \textit{the function } \widetilde{u} \textit{ is bounded in the half-plane } \mathbb{C}_\mu = \{s \in \mathbb{C} : \mathrm{Re}(s) \geq \mu\}, \tag{1.35}$$

$$\sup_{\mathrm{Im}(s)\in\mathbb{R}} |\widetilde{u}(s)| \to 0 \qquad \textit{for } \mathrm{Re}(s) \to +\infty. \tag{1.36}$$

(The latter statement is reminiscent of the Riemann-Lebesgue Theorem of the Fourier transform.)

Proof. For any $s \in \mathbb{C}_\mu$, $\mathrm{Re}[-(s-\mu)t] < 0$ whence $|e^{-(s-\mu)t}| \leq 1$. Therefore

$$|\widetilde{u}(s)| = \Big|\int_{\mathbb{R}} e^{-st}u(t)\,dt\Big| \leq \int_{\mathbb{R}} |e^{-\mu t}||e^{-(s-\mu)t}|u(t)|\,dt \leq \int_{\mathbb{R}} e^{-\mu t}|u(t)|\,dt \qquad \forall s \in \mathbb{C}_\mu, \tag{1.37}$$

and (1.35) follows.

By taking the supremum of this inequality w.r.t. $\mathrm{Im}(s) \in \mathbb{R}$ and then passing to the limit as $\mathrm{Re}(s) \to +\infty$, (1.36) follows by dominated convergence. □

Remarks 1.10 (i) The transformed function $\widetilde{u}$ need not be bounded in the whole convergence half-plane $\mathbb{C}_{\lambda(u)}$. E.g., $\widetilde{H}(s) = 1/s$ is a counterexample.

(ii) If the Laplace integral $\widetilde{u}(s)$ absolutely converges for some $s \in \mathbb{C}$, then the same occurs for $s+iy$ for any $y \in \mathbb{R}$. Therefore the set of absolute convergence is either of the form $\{s \in \mathbb{C} : \mathrm{Re}(s) > a\}$ or $\{s \in \mathbb{C} : \mathrm{Re}(s) \geq a\}$ for some $a \in\,]-\infty, +\infty[$, or it is the whole $\mathbb{C}$. In the first case $\widetilde{u}(s)$ is bounded, in the second one it is not so, if $\mathbb{C}_{\lambda(u)}$ is a proper subset of $\mathbb{C}$. Note that in either case the half-plane of absolute convergence is of the form $\{s \in \mathbb{C} : \mathrm{Re}(s) > a\}$ (or the whole $\mathbb{C}$).

(iii) In some cases $\widetilde{u}$ can be extended to a holomorphic function defined on a larger domain than $\mathbb{C}_{\lambda(u)}$. For instance, the transform $\widetilde{H}(s)$ $(= 1/s)$ of the Heaviside function H is only defined for $\mathrm{Re}(s) > \lambda(H) = 0$, although the function $s \mapsto 1/s$ is holomorphic in $\mathbb{C} \setminus \{0\}$.

(iv) The theorem of holomorphism is one of the main differences between the properties of the Laplace and Fourier transforms. Fourier transformed functions depend on a real variable, and may not even be differentiable on $\mathbb{R}$. On the other hand, Laplace-transformed functions depend on a a complex variable, and for them first-order differentiability entails infinite differentiability.

(v) In several cases one can perform a computation for real s, and then extend the result to the convergence half-plane by the theorem of holomorphism. □

2 Further Properties of the Laplace Transform

The Laplace transform fulfills differentiation rules that have some analogies with those of the Fourier transform. However, here the initial value of the input function u plays a role, and this makes the Laplace transform especially useful for the study of initial value problems.

Differentiation. For any function v and any $n \in \mathbb{N}$, we shall denote by $v', v'', ..., v^{(n)}$ the derivatives almost everywhere, if they exist.

Proposition 2.1 *(Differentiation–I) For any $u \in D_{\mathcal{L}}$,*

$$\lambda(t^n u(t)) = \lambda(u), \qquad [\mathcal{L}(u)]^{(n)}(s) = (-1)^n [\mathcal{L}(t^n u(t))](s) \qquad \forall s \in \mathbb{C}_{\lambda(u)}, \forall n \in \mathbb{N}. \tag{2.1}$$

Outline of the proof. As we saw, the multiplication of the input function by t does not modify the convergence abscissa. As $\mathcal{L}(u)$ is holomorphic, it is indefinitely differentiable. By dominated convergence, one can differentiate under the Laplace integral, see e.g. [Gilardi p. 396]. One thus gets $\mathcal{L}(u)' = -\mathcal{L}(tu(t))$. This is then easily extended to higher-order derivatives by recurrence. □

• **Theorem 2.2** *(Differentiation–II) (i) For any $u \in D_{\mathcal{L}}$ that is absolutely continuous in* $]0, +\infty[$, *if $u' \in D_{\mathcal{L}}$ and if there exists $u(0^+) := \lim_{t \to 0^+} u(t) \in \mathbb{C}$, then*

$$[\mathcal{L}(u')](s) = s[\mathcal{L}(u)](s) - u(0^+) \qquad \forall s \in \mathbb{C}_{\lambda(u)} \cap \mathbb{C}_{\lambda(u')}. \tag{2.2}$$

(ii) More generally, let $u \in D_{\mathcal{L}}$ and $u, u', ..., u^{(m-1)}$ be absolutely continuous in $]0, +\infty[$ *for an integer $m > 1$. If $u', ..., u^{(m)} \in D_{\mathcal{L}}$ and if the limits $u(0^+), ..., u^{(m-1)}(0^+)$ exist in $\mathbb{C}$, then*

$$[\mathcal{L}(u^{(m)})](s) = s^m [\mathcal{L}(u)](s) - \sum_{n=0}^{m-1} s^{m-n-1} u^{(n)}(0^+) \qquad \forall s \in \mathbb{C}_{\lambda(u)} \cap ... \cap \mathbb{C}_{\lambda(u^{(m)})}. \tag{2.3}$$

We assume absolute continuity just for $t > 0$, in order to allow the input function u (and in part (ii) also $u', ..., u^{(m-1)}$) to have a jump at $t = 0$. In this way we shall be able to prescribe nontrivial initial conditions for initial-value problems for ODEs.

Proof. (i) As $e^{-st}u(t) \in L^1_t$ whenever $\mathrm{Re}(s) > \lambda(u)$, and as this function is continuous, there exists a sequence $\{t_n\}$ such that $t_n \to +\infty$ and $e^{-st_n}u(t_n) \to 0$ as $n \to +\infty$. By partial integration, we then get

$$\begin{aligned}
[\mathcal{L}(u')](s) &= \int_0^{+\infty} e^{-st} u'(t)\,dt = \lim_{n\to\infty} \int_0^{t_n} e^{-st} u'(t)\,dt \\
&= \lim_{n\to\infty} \left\{ \int_0^{t_n} s e^{-st} u(t)\,dt + e^{-st_n} u(t_n) - u(0^+) \right\} \\
&= s \int_0^{+\infty} e^{-st} u(t)\,dt - u(0^+) \qquad \forall s \in \mathbb{C}_{\lambda(u)} \cap \mathbb{C}_{\lambda(u')}. \qquad \square
\end{aligned} \tag{2.4}$$

(ii) For instance, for $m = 2$

$$[\mathcal{L}(u'')](s) = s[\mathcal{L}(u')](s) - u'(0^+) = s^2[\mathcal{L}(u)](s) - su(0^+) - u'(0^+),$$

for all $s \in \mathbb{C}_{\lambda(u)} \cap \mathbb{C}_{\lambda(u')} \cap \mathbb{C}_{\lambda(u'')}$. This simple procedure is easily extended to prove (2.3) via a recurrence argument which is left to the Reader. □

Remark 2.3 In (3.22) we shall give an example of $u \in D_{\mathcal{L}}$ with $\lambda(u) < \lambda(u')$. □

Proposition 2.4 *(Integration) If $u, u(t)/t \in D_{\mathcal{L}}$ then $\lambda(u(t)/t) = \lambda(u)$ and* [3]

$$[\mathcal{L}(u(t)/t)](s) = \lim_{\mathbb{R}\ni\sigma\to+\infty} \int_s^\sigma [\mathcal{L}(u)](r)\,dr \qquad \forall s \in \mathbb{C}_{\lambda(u)}. \tag{2.5}$$

Proof. By applying the differentiation theorem to $v(t) := u(t)/t$, we have

$$\lambda(u(t)/t) = \lambda(u), \qquad \widetilde{v}' = -\widetilde{tv(t)} = -\widetilde{u},$$

whence by integration

$$\widetilde{v}(s) = \widetilde{v}(\sigma) + \int_s^\sigma \widetilde{u}(r)\,dr \qquad \forall s \in \mathbb{C}_{\lambda(u)}, \forall \sigma > \lambda(u).$$

Passing to the limit as $\sigma \to +\infty$ and recalling (1.36), we get (2.5). □

Remark 2.5 The limit (2.5) coincides with the generalized integral of $\mathcal{L}(u)$ between s and $+\infty$ (here $+\infty$ is the limit of $(x, 0)$ as $x \to +\infty$...); this integral does not depend on the integration path, because $\mathcal{L}(u)$ is holomorphic. Anyway this generalized integral need not converge absolutely, thus it may not be a Lebesgue integral. □

Inversion of the Laplace transform. The following theorem provides an explicit formula for the antitransform. Let us first say that a function $u \in D_{\mathcal{L}}$ is of *exponential order* if

$$\exists \alpha \in \mathbb{R}, \exists M > 0: \qquad |u(t)| \le M e^{\alpha t} \text{ for a.e. } t > 0. \tag{2.6}$$

Let us denote by $\mathcal{O}$ this subset of $D_{\mathcal{L}}$. It includes most of the functions that occur in applications, and is strictly contained in $D_{\mathcal{L}}$. For instance, $u(t) = t^{-1/2}H(t) \in D_{\mathcal{L}} \setminus \mathcal{O}$. It is easy to see that

$$\lambda(u) = \inf\{\alpha \in \mathbb{R} : (2.6) \text{ is fulfilled}\} \qquad \forall u \in \mathcal{O}. \tag{2.7}$$

• **Theorem 2.6** *(Riemann-Fourier) For any $u \in \mathcal{O}$, denoting by $\widetilde{u}$ its Laplace transform,*

$$u(t) = \frac{1}{2\pi i} \text{P.V.} \int_{x+i\mathbb{R}} e^{st}\widetilde{u}(s)\,ds \qquad \textit{for a.e. } t \in \mathbb{R}, \forall x > \lambda(u). \tag{2.8}$$

This principal value is the limit as $R \to +\infty$ of the integral along a path in $\mathbb{C}_{\lambda(u)}$ from $s = x - iR$ to $s = x + iR$. Setting $s = x + iy$, the Riemann-Fourier inversion formula can also be rewritten as

$$u(t) = \frac{1}{2\pi} \lim_{R\to+\infty} \int_{-R}^{R} e^{(x+iy)t}\widetilde{u}(x+iy)\,dy \qquad \text{for a.e. } t \in \mathbb{R}, \forall x > \lambda(u). \tag{2.9}$$

Proof. We shall reduce the Laplace transform to the Fourier transform, see (1.16), and then invert the latter. Let us select any $x > \lambda(u)$ and set

$$\varphi_x(y) := \widetilde{u}(x+iy) = \int_{\mathbb{R}} e^{-(x+iy)t}u(t)\,dt \qquad \forall y \in \mathbb{R}, \tag{2.10}$$

[3] Here we mean that σ is real; anyway, because of the holomorphism of $\mathcal{L}(u)$, this limit can also be taken as σ varies along a complex path in $\mathbb{C}_{\lambda(u)}$.

that is, $\varphi_x(y) = \sqrt{2\pi}[\mathcal{F}(e^{-xt}u(t))](y)$, as in (1.15). By (2.7), we can assume that $\lambda(u) \leq \alpha < x$, so that

$$|e^{-(x+iy)t}u(t)| = |e^{-xt}u(t)| \overset{(2.6)}{\leq} Me^{(\alpha-x)t}H(t) \in L^2.$$

The Fourier transform $\mathcal{F}(e^{-xt}u(t))$ can thus be understood not only in the sense of L^1 (i.e., as a Lebesgue integral) but also in the sense of L^2, thus as a principal value, see (**??**). [Fourier chapter] The theorem of inversion of the Fourier transform in L^2 then yields

$$e^{-xt}u(t) = \frac{1}{\sqrt{2\pi}}\big[\mathcal{F}^{-1}(\varphi_x)\big](t) = \frac{1}{2\pi}\text{P.V.}\int_{\mathbb{R}} e^{ity}\varphi_x(y)dy \qquad \text{for a.e. } t \in \mathbb{R}, \forall x > \lambda(u), \tag{2.11}$$

that is, (2.8) by (2.10). □

An exercise in complex calculus. By the foregoing argument, the principal value occurring in (2.8) does not depend on the selected $x > \lambda(u)$. Here we justify this under the simplifying assumption that

$$\exists\, C, a > 0 : \forall s \in \mathbb{C}_\lambda(u) \qquad |\widetilde{u}(s)| \leq C|s|^{-a}. \tag{2.12}$$

For any x_1, x_2 with $\lambda(u) < x_1 < x_2$, because of Proposition 1.8 the function $s \mapsto e^{st}\widetilde{u}(s)$ is holomorphic in the strip of the complex plane that is comprised between the straight lines $x_1 + i\mathbb{R}$ and $x_2 + i\mathbb{R}$. For any $R > 0$ let us denote by B_R the boundary of the rectangle of vertices $x_1 - iR, x_1 + iR, x_2 + iR, x_2 - iR$. As the funtion $s \mapsto e^{st}\widetilde{u}(s)$ is holomorphic in this rectangle, by the classical Cauchy integral theorem,

$$\int_{B_R} e^{st}\widetilde{u}(s)ds = 0 \qquad \forall R > 0. \tag{2.13}$$

Because of (2.12) the modulus of the integral along the horizontal segment $y = R$ is dominated by

$$\int_{x_1}^{x_2} e^{xt}|\widetilde{u}(x + iR)|\, dx \leq Ce^{x_2 t}|x_1 + iR|^{-a}(x_2 - x_1),$$

which vanishes as $R \to +\infty$. The same holds for the modulus of the integral along the horizontal segment $y = -R$. (2.13) then yields

$$\int_{x_1-iR}^{x_1+iR} e^{st}\widetilde{u}(s)ds - \int_{x_2-iR}^{x_2+iR} e^{st}\widetilde{u}(s)ds \to 0 \qquad \text{as } R \to +\infty.$$

As both these integrals converge in the sense of the principal value as $R \to +\infty$, we conclude that

$$\frac{1}{2\pi i}\text{P.V.}\int_{x_1+i\mathbb{R}} e^{st}\widetilde{u}(s)\, ds = \frac{1}{2\pi i}\text{P.V.}\int_{x_2+i\mathbb{R}} e^{st}\widetilde{u}(s)\, ds.$$

Remarks 2.7 (i) By the inversion theorem, $\widetilde{u} = 0$ only if $u = 0$. The Laplace transform is thus injective on the whole $D_{\mathcal{L}}$.

(ii) The computation of the Riemann-Fourier formula often requires the evaluation of *contour integrals* in the complex plane, and this may also involve tools of complex analysis like the Jordan lemma, the residue theorem, and so on. In several cases it is simpler to antitransform $\widetilde{u}$ by expanding it in *simple fractions,* and then use transform tables *backward,* as we shall see in Section 4. □

Antitransformability. Unfortunately, the Riemann-Fourier Theorem deals with transformable rather than transformed functions. The problem of characterizing the image set $\mathcal{L}(D_{\mathcal{L}})$ is nontrivial, and here we just provide a sufficient condition for the existence of the inverse transform.

Theorem 2.8 *(Antitransformability–I) The restriction of a function U of complex variable to a suitable half-plane* [4] *is the Laplace transform of some function $u \in D_{\mathcal{L}}$ if* [5]

$$\exists \lambda \in \mathbb{R}: \ U \text{ is holomorphic for } \operatorname{Re}(s) > \lambda, \tag{2.14}$$

$$\exists \alpha > 1: \ \sup_{y\in\mathbb{R}} |x+iy|^{\alpha} |U(x+iy)| \to 0 \quad \text{as } x \to +\infty. \tag{2.15}$$

Moreover, in this case u is continuous. []

This implication cannot be inverted. In particular note that the sufficient condition (2.15) is stronger than the necessary condition (1.36). For instance, for any $a > 0$, $U(s) = e^{-as}$ is the transform of no function. [6]

Regrettably, the condition (2.15) fails for the function $U(s) = 1/s$, which however is the transform of the Heaviside function for $\operatorname{Re}(s) > 0$. This example and the theorem above entail the next statement, which is often applied because of the ubiquity of the Heaviside function.

Corollary 2.9 *Let U be a complex function of complex variable which fulfills* (2.14). *For any $a \in \mathbb{C}$, the function $V(s) = U(s) + a/s$ is also antitransformable, and $\mathcal{L}^{-1}(V) = \mathcal{L}^{-1}(U) + aH$.*

Corollary 2.10 *The restriction of a function U of complex variable to a suitable half-plane is the Laplace transform of some function $u \in D_{\mathcal{L}}$ if it fulfills* (2.14) *and*

$$\exists \alpha > 1 : |s|^{\alpha} |U(s)| \to 0 \quad \text{as } |s| \to +\infty. \tag{2.16}$$

In this case the Riemann-Fourier formula (2.9) *can be rewritten as*

$$u(t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{(x+iy)t} U(x+iy)\, dy \qquad \text{for a.e. } t \in \mathbb{R}, \forall x > \lambda(u). \tag{2.17}$$

Proof. The first statement follows from the Antitransformability Theorem 2.8, since the condition (2.16) entails (2.15). Because of (2.16) the principal value (2.9) is actually a Lebesgue integral. □

The next theorem is an example of so-called *Tauberian theorem.* These results provide information on the original function u on the basis of the transformed function $\widetilde{u}$, without inverting the Laplace transformation.

*** Theorem 2.11** *(Final and initial values) For any $u \in D_{\mathcal{L}}$,*

$$\exists u(+\infty) \in \mathbb{C} \quad \Rightarrow \quad \lambda(u) \le 0, \quad s\widetilde{u}(s) \to u(+\infty) \ \text{ as } \ s \to 0, \ ^{7} \tag{2.18}$$

$$\exists u(0^+) \in \mathbb{C} \quad \Rightarrow \quad \sup_{\operatorname{Im}(s)\in\mathbb{R}} |s\widetilde{u}(s) - u(0^+)| \to 0 \ \text{ as } \ \operatorname{Re}(s) \to +\infty. \tag{2.19}$$

[4] Of course, we refer to half-planes of the form $\{s \in \mathbb{C} : \operatorname{Re}(s) > \lambda\}$, for some $\lambda \in [-\infty, +\infty[$.

[5] (2.15) can be restated as $\exists \alpha > 1 : \forall \varepsilon > 0, \exists L > 0$ such that if $\operatorname{Re}(s) > L$ then $|s|^{\alpha}|U(s)| \le \varepsilon$.
This means that, for a suitable $\alpha > 1$, $|s|^{\alpha} U(s) \to 0$ as $\operatorname{Re}(s) \to +\infty$, uniformly w.r.t. $\operatorname{Im}(s)$. The weaker necessary condition (1.36) corresponds to $\alpha = 0$.

[6] We shall see that it is the transform the singular distribution δ_a.

If $\lambda(u) = 0$ then in (2.18) it is understood that $s \to 0$ from the half-plane of complex numbers with positive real part.

(2.19) means that, as $\mathrm{Re}(s) \to +\infty$, $s\widetilde{u}(s) \to u(0^+)$ uniformly w.r.t. $\mathrm{Im}(s)$.

*** Proof.** This result (actually, a much more general statement) is proved e.g. by [Gilardi p. 429]. In order to simplify the argument, here we assume that

$$u \text{ is absolutely continuous in }]0, +\infty[, \quad u' \in D_{\mathcal{L}}, \quad \exists u(+\infty), u(0^+) \in \mathbb{C}. \tag{2.20}$$

This entails that u is bounded, whence $\lambda(u) \le 0$. By dominated convergence,

$$\int_0^{+\infty} u'(t)\, dt = \lim_{s\to 0} \int_0^{+\infty} e^{-st} u'(t)\, dt = \lim_{s\to 0} \widetilde{u'}(s) \overset{(2.2)}{=} \lim_{s\to 0} s\widetilde{u}(s) - u(0^+). \tag{2.21}$$

On the other hand

$$\int_0^{+\infty} u'(\tau)\, d\tau = \lim_{t\to+\infty} \int_0^{t} u'(\tau)\, d\tau = u(+\infty) - u(0^+), \tag{2.22}$$

and by comparing these two identities we get (2.18).

To prove (2.19) it suffices to notice that

$$\lim_{\mathrm{Re}(s)\to+\infty} s\widetilde{u}(s) - u(0^+) \overset{(2.2)}{=} \lim_{\mathrm{Re}(s)\to+\infty} \widetilde{u'}(s) \overset{(1.36)}{=} 0. \qquad \square \tag{2.23}$$

Remarks 2.12 (i) The final-value theorem cannot be inverted: the existence of $\lim_{s\to 0} s\widetilde{u}(s)$ in $\mathbb{C}$ does not entail that of $u(+\infty)$. For instance $u(t) = (\sin t)H(t)$ does not converge as $t \to +\infty$, although $s\widetilde{u}(s) = s/(s^2+1) \to 0$ as $s \to 0$.

(ii) The initial-value theorem also cannot be inverted: the existence of $\lim_{\mathrm{Re}(s)\to+\infty} s\widetilde{u}(s)$ in $\mathbb{C}$ (uniformly w.r.t. $\mathrm{Im}(s)$) does not entail that of $u(0^+)$. A counterexample is provided for instance by [Doetsch 144]; we omit it, as it is less simple than the previous one. $\square$

Theorem 2.13 *(Titchmarsch) For any $u, v \in D_{\mathcal{L}}$, if $\widetilde{u}\widetilde{v} = 0$ then either $u = 0$ or $v = 0$.*

We refer to [Amerio 218] for the proof of this classical result.

3 Laplace Transform of Distributions

We saw that the Laplace transform extends the Fourier transform defined in L^1. As the latter transform can be extended to the space $\mathcal{S}'$ of tempered distributions, one may wonder whether the Laplace transform can also be extended to some space of distributions. We shall answer in the affirmative, under suitable restrictions.

Extended Laplace transform $\widehat{\mathcal{L}}$. Let us define the class $D_{\widehat{\mathcal{L}}}$ of transformable distributions with convergence abscissa $\widehat{\lambda}$: [8]

$$D_{\widehat{\mathcal{L}}} := \{T \in \mathcal{D}' : \mathrm{supp}(T) \subset \mathbb{R}^+, \exists x \in \mathbb{R} : e^{-xt}T \in \mathcal{S}'_t\}, \tag{3.1}$$

$$\widehat{\lambda}(T) := \inf \{x \in \mathbb{R} : e^{-xt}T \in \mathcal{S}'_t\} \ (\in [-\infty, +\infty[) \qquad \forall T \in D_{\widehat{\mathcal{L}}}, \tag{3.2}$$

$$\mathbb{C}_{\widehat{\lambda}(T)} := \{s \in \mathbb{C} : \ \mathrm{Re}(s) > \widehat{\lambda}(T)\} \qquad \forall T \in D_{\widehat{\mathcal{L}}}. \tag{3.3}$$

[8] As we know, for any $T \in \mathcal{S}'$, $\mathrm{supp}(T) \subset \mathbb{R}^+$ means that $\langle T, v\rangle = 0$ for any $v \in \mathcal{D}$ such that $\mathrm{supp}(v) \subset]-\infty, 0[$. We use the notation $\mathcal{S}'_t$ to indicate that t is the independent variable, whereas x is just a parameter.

Here we do not speak of *absolute* convergence, which is meaningless for duality pairings.

We are tempted to define the Laplace transform as in (1.11), just replacing the integral by a suitable duality pairing $\langle\cdot,\cdot\rangle$:

$$[\widehat{\mathcal{L}}(T)](s) = \langle T, e^{-st}\rangle \qquad \forall s \in \mathbb{C}_{\widehat{\lambda}(T)}, \forall T \in D_{\widehat{\mathcal{L}}}. \tag{3.4}$$

But here we cannot use the duality between $\mathcal{D}'$ and $\mathcal{D}$, as the support of e^{-st} is not compact. We might use the pairing between $\mathcal{E}'$ and $\mathcal{E}$, but dealing just with compactly supported distributions would be too restrictive. We cannot use the pairing between $\mathcal{S}'$ and $\mathcal{S}$, since e^{-st} does not decay as $t \to +\infty$ whenever $\mathrm{Re}(s) \leq 0$, and we do not intend to exclude $\widehat{\lambda}(T) < 0$. Hiowever, we do not expect to encounter any substantial difficulty for $t \to -\infty$, since the support of T is confined to $\mathbb{R}^+$.

We then use the following construction. The idea is to select any $x \in]\widehat{\lambda}(T), \mathrm{Re}(s)[$, and to rewrite (3.4) as

$$[\widehat{\mathcal{L}}(T)](s) = \langle e^{-xt}T, e^{(x-s)t}\rangle \qquad \forall s \in \mathbb{C}_{\widehat{\lambda}(T)}, \forall T \in D_{\widehat{\mathcal{L}}}, \tag{3.5}$$

since $e^{-xt}T \in \mathcal{S}'$ by (3.1), and $e^{(x-s)t}$ decays as $t \to +\infty$, since $\mathrm{Re}(x-s) < 0$. But $e^{(x-s)t} \notin \mathcal{S}$, because of its behaviour as $t \to -\infty$, so that we cannot interpret (3.5) in the duality between $\mathcal{S}'$ and $\mathcal{S}$. However, as T is supported on $\mathbb{R}^+$, we overcome this difficulty via the following expedient. We fix any function $\zeta \in C^\infty$ such that

$$\zeta = 0 \quad \text{in }]-\infty, a[, \text{ for some } a < 0, \qquad \zeta = 1 \quad \text{in } \mathbb{R}^+, \tag{3.6}$$

so that $e^{(x-s)t}\zeta(t) \in \mathcal{S}$. We then set

$$[\widehat{\mathcal{L}}(T)](s) := {}_{\mathcal{S}'}\langle e^{-xt}T, e^{(x-s)t}\zeta(t)\rangle_{\mathcal{S}} \qquad \forall s \in \mathbb{C}_{\widehat{\lambda}(T)}, \forall T \in D_{\widehat{\mathcal{L}}}. \tag{3.7}$$

This value does not depend on the choice of $x \in]\widehat{\lambda}(T), \mathrm{Re}(s)[$ and of the auxiliary function ζ. This definition is thus informally equivalent to (3.4), which might be regarded just as a short-writing of the rigorous formula (3.7).

We shall use the notation $\widetilde{T}$ also for any $T \in D_{\widehat{\mathcal{L}}}$.

Any sequence $\{T_n\} \subset D_{\widehat{\mathcal{L}}}$ is said to converge to $T \in D_{\widehat{\mathcal{L}}}$ in the sense of $D_{\widehat{\mathcal{L}}}$ iff

(i) there exists $x \in \mathbb{R}$ such that $x \geq \lambda(T)$ and $x \geq \lambda(T_n)$ for any n,

(ii) $e^{-xt}T_n(t) \to e^{-xt}T$ in $\mathcal{S}'$.

The next statement is easily checked.

Proposition 3.1 *For any sequence $\{T_n\}$ in $D_{\widehat{\mathcal{L}}}$*

$$T_n \to T \ \text{ in } D_{\widehat{\mathcal{L}}} \quad \Rightarrow \quad \widetilde{T_n}(s) \to \widetilde{T}(s) \quad \forall s \in \mathbb{C}_{\widehat{\lambda}(T)}. \tag{3.8}$$

Here is an important example:

$$\delta_a \in D_{\widehat{\mathcal{L}}}, \qquad \widetilde{\delta_a}(s) = e^{-as} \qquad \forall s \in \mathbb{C}, \forall a \geq 0; \tag{3.9}$$

in particular, $\widetilde{\delta_0}(s) = 1$. More generally, any causal compactly supported distribution is Laplace transformable.

As $L^1 \subset \mathcal{S}'$, the next statement is easily proved. As usual, we identify any function $u \in L^1_{\mathrm{loc}}$ with the corresponding distribution T_u.

Proposition 3.2 *The transform $\widehat{\mathcal{L}}$ of distributions extends that of functions (denoted by $\mathcal{L}$):*

$$D_{\mathcal{L}} \subset D_{\widehat{\mathcal{L}}}, \tag{3.10}$$

$$\widehat{\lambda}(u) \leq \lambda(u) \quad \textit{i.e.} \quad \mathbb{C}_{\lambda(u)} \subset \mathbb{C}_{\widehat{\lambda}(u)} \qquad \forall u \in D_{\mathcal{L}}, \tag{3.11}$$

$$\widetilde{u}|_{\mathbb{C}_{\lambda(u)}} = \mathcal{L}(u) \qquad \forall u \in D_{\mathcal{L}}. \tag{3.12}$$

Note that here the use of the duality pairing instead of the (Lebesgue) integral overcomes the restriction of *absolute* convergence. We shall also see that:

$$D_{\widehat{\mathcal{L}}} \cap L^1_{\text{loc}} \not\subset D_{\mathcal{L}}, \quad \text{thus } \mathcal{L} \neq \bar{\mathcal{L}}\big|_{D_{\widehat{\mathcal{L}}} \cap L^1_{\text{loc}}}, \tag{3.13}$$

$$\exists u \in D_{\mathcal{L}} : \widehat{\lambda}(u) < \lambda(u), \quad \text{thus } \mathcal{L} \neq \bar{\mathcal{L}}\big|_{D_{\mathcal{L}}}, \text{ despite of (3.12).} \tag{3.14}$$

* **Remark 3.3** Similarly to what we saw for $D_{\mathcal{L}}$, the linear space of transformable distributions $D_{\widehat{\mathcal{L}}}$ can be equipped with the following notion of sequential convergence: a sequence $\{u_n\}$ of $D_{\widehat{\mathcal{L}}}$ is said to converge to some $u \in D_{\widehat{\mathcal{L}}}$ in this space iff
(1) $\widehat{\lambda}(u) = \liminf_{n\to+\infty} \widehat{\lambda}(u_n) > -\infty$,
(2) $e^{-xt}u_n(t) \to e^{-xt}u(t)$ in $\mathcal{S}'$, for any $x > \widehat{\lambda}(u)$.
It is easy to see that this entails $\widehat{\mathcal{L}}(u_n) \to \widehat{\mathcal{L}}(u)$ pointwise in $\mathbb{C}_{\widehat{\lambda}(u)}$, for any $u \in D_{\widehat{\mathcal{L}}}$.

Basic properties. Theorems 1.2, 1.3, 1.5, 1.6 and 1.8 can be extended to $\bar{\mathcal{L}}$. In particular, the Laplace transform of distributions is holomorphic. The Riemann-Fourier inversion formula can also be extended to transformable distributions under hypotheses that here we do not display. On the other hand, (1.35) and (1.36) do not take over to distributions: e.g., $\widetilde{D\delta_0}(s) = s$ (see (3.19) ahead).

Next we deal with the transform of the distributional derivative, that we denote by D, and prove a simpler formula than for functions, under weaker hypotheses.

• **Theorem 3.4** *(Distributional differentiation) For any $T \in D_{\widehat{\mathcal{L}}}$,*

$$DT \in D_{\widehat{\mathcal{L}}}, \qquad \widehat{\lambda}(DT) \leq \widehat{\lambda}(T), \qquad [\widehat{\mathcal{L}}(DT)](s) = s[\widehat{\mathcal{L}}(T)](s) \quad \forall s \in \mathbb{C}_{\widehat{\lambda}(T)}. \tag{3.15}$$

More generally, for any $k \in \mathbb{N}$,

$$D^kT \in D_{\widehat{\mathcal{L}}}, \qquad \widehat{\lambda}(D^kT) \leq \widehat{\lambda}(T), \qquad [\widehat{\mathcal{L}}(D^kT)](s) = s^k[\widehat{\mathcal{L}}(T)](s) \quad \forall s \in \mathbb{C}_{\widehat{\lambda}(T)}. \tag{3.16}$$

Proof. One might be tempted to use the formula (3.5), getting $\langle DT, e^{-st}\rangle = \langle T, e^{-st}\rangle$. But this would be just heuristic, because here the duality pairing is meaningless, as we pointed out above.

On a more firm ground, let us fix any function $\zeta \in C^\infty$ as in (3.6) and any $x \in]\widehat{\lambda}(T), \text{Re}(s)[$. By (3.5),

$$\begin{aligned}
&\langle e^{-xt}DT, e^{(x-s)t}\zeta(t)\rangle = \langle D(e^{-xt}T) + xe^{-xt}T, e^{(x-s)t}\zeta(t)\rangle \\
&= -\langle e^{-xt}T, D[e^{(x-s)t}\zeta(t)]\rangle + \langle xe^{-xt}T, e^{(x-s)t}\zeta(t)\rangle \\
&= \langle e^{-xt}T, (s-x)e^{(x-s)t}\zeta(t)\rangle - \langle e^{-xt}T, e^{(x-s)t}\zeta'(t)\rangle + \langle xe^{-xt}T, e^{(x-s)t}\zeta(t)\rangle \\
&= \langle e^{-xt}T, se^{(x-s)t}\zeta(t)\rangle - \langle e^{-xt}T, e^{(x-s)t}\zeta'(t)\rangle = s[\widehat{\mathcal{L}}(T)](s) \qquad \forall s \in \mathbb{C}_{\widehat{\lambda}(T)}
\end{aligned} \tag{3.17}$$

(the term in ζ' vanishes since its support does not intersect that of T). (3.15) is thus established. (3.16) then follows by recurrence. □

For instance,

$$[\widehat{\mathcal{L}}(D^k \delta_a)](s) = s^k [\widehat{\mathcal{L}}(\delta_a)](s) \overset{(3.9)}{=} s^k e^{-as} \qquad \forall s \in \mathbb{C}, \forall a \geq 0, \forall k \in \mathbb{N}, \tag{3.18}$$

in particular

$$[\widehat{\mathcal{L}}(D^k \delta_0)](s) = s^k \qquad \forall s \in \mathbb{C}, \forall k \in \mathbb{N}. \tag{3.19}$$

Let us now assume the hypotheses of part (i) of Theorem 2.2, that is,

$$u \in D_{\mathcal{L}}, \quad u \text{ is absolutely continuous in }]0, +\infty[, \quad u' \in D_{\mathcal{L}}, \quad \exists u(0^+) := \lim_{t \to 0^+} u(t) \in \mathbb{C}. \tag{3.20}$$

Hence $Du = u' + u(0^+)\delta_0$ in $\mathcal{D}'$, and Theorem 3.4 yields

$$s[\widehat{\mathcal{L}}(u)](s) \overset{(3.15)}{=} \widehat{\mathcal{L}}(Du) = \widehat{\mathcal{L}}(u' + u(0^+)\delta_0) = \mathcal{L}(u') + u(0^+) \qquad \text{in } \mathbb{C}_{\tilde{\lambda}(u)}. \tag{3.21}$$

We thus retrieve (2.2). Under the assumptions of part (ii) of Theorem 2.2, (2.3) can similarly be retrieved.

Two counterexamples. (i) Next we exhibit a function $u \in D_{\mathcal{L}}$ with $\widehat{\lambda}(u) < \lambda(u)$. Let us set

$$v(t) = [\sin(e^t - 1)]H(t) \in D_{\mathcal{L}}, \tag{3.22}$$

whence

$$u(t) := Dv(t) = e^t [\cos(e^t - 1)]H(t) \in D_{\mathcal{L}}.\,^{9} \tag{3.23}$$

Here $\lambda(v) = 0$ and $\lambda(u) = 1$. As

$$\widehat{\lambda}(u) \overset{(3.15)}{\leq} \widehat{\lambda}(v) \overset{(3.11)}{\leq} \lambda(v) < \lambda(u), \tag{3.24}$$

we conclude that in this case $\widehat{\lambda}(u) < \lambda(u)$, as we anticipated in (3.14).

By this example, the abscissa of *absolute* convergence of the almost everywhere derivative of an absolutely continuous transformable function v can thus be larger than that of v. This is at variance with what we saw for the distributional derivative of the transform of distributions.

(ii) Let us consider the function

$$v(t) = [\sin(e^{t^2} - 1)]H(t) \in D_{\mathcal{L}} (\subset D_{\widehat{\mathcal{L}}}), \tag{3.25}$$

whence

$$u(t) = Dv(t) = 2te^{t^2} [\cos(e^{t^2} - 1)]H(t). \tag{3.26}$$

By Theorem 3.4, $u \in D_{\widehat{\mathcal{L}}} \cap L^1_{\text{loc}}$ despite of the presence of the exponential. But $u \notin D_{\mathcal{L}}$, as the Laplace integral of this function does not converge absolutely for any s. This proves (3.13).

[9] In passing note that $u \in \mathcal{S}'$, despite of the occurrence of the exponential, since it is the derivative of a function of $\mathcal{S}'$. Actually u has no genuine exponential growth, because of the oscillating factor.

Theorem 3.5 *(Antitransformability–II) The restriction of a function U of complex variable to a suitable half-plane is the Laplace transform of some distribution $T \in D_{\widehat{\mathcal{L}}}$ iff*

$$\exists\lambda \in \mathbb{R} : \ U \text{ is holomorphic for } \operatorname{Re}(s) > \lambda, \tag{3.27}$$

$$\exists M > 0, \exists a \geq 0 : \forall s \in \mathbb{C}_\lambda, \ |U(s)| \leq M(1 + |s|)^a. \qquad [] \tag{3.28}$$

This theorem characterizes the Laplace transform of distributions, whereas Theorem 2.8 just provided a sufficient condition for a function to be the transform of a function. The Reader will notice how weaker is the condition (3.28) than (2.15).

4 Laplace Antitransform of Rational Functions

Henceforth we shall drop the bar, and write $\mathcal{L}$, $D_{\mathcal{L}}$, λ, instead of $\widehat{\mathcal{L}}$, $D_{\widehat{\mathcal{L}}}$, $\widehat{\lambda}$. The context will make clear whether we refer to the Laplace transform of functions or of distributions.

Let F be a *rational function* of s, namely the quotient of two algebraic polynomials with complex coefficients:

$$F(s) = \frac{P(s)}{Q(s)} = \frac{\sum_{j=0}^{m} a_j s^j}{\sum_{\ell=0}^{n} b_\ell s^\ell} \qquad (a_m, b_n \neq 0, \ m, n \in \mathbb{N}). \tag{4.1}$$

It is known that if $m \geq n$ then there exist $c_0, ..., c_{m-n-1} \in \mathbb{C}$, and a polynomial $\widetilde{P}(s) = \sum_{j=0}^{\widetilde{m}} \widetilde{a}_j s^j$ of degree $\widetilde{m} < n$ such that, setting $c_{m-n} = a_m/b_n (\neq 0)$,

$$F(s) = \sum_{k=0}^{m-n} c_k s^k + \frac{\widetilde{P}(s)}{Q(s)} =: G(s) + R(s) \qquad \text{for } \operatorname{Re}(s) > \lambda. \tag{4.2}$$

The coefficients of $G(s)$ and $\widetilde{P}(s)$ can be determined via the identity $P(s) = G(s)Q(s) + \widetilde{P}(s)$, that is,

$$\sum_{j=0}^{m} a_j s^j = \left(\sum_{k=0}^{m-n} c_k s^k\right)\left(\sum_{\ell=0}^{n} b_\ell s^\ell\right) + \sum_{j=0}^{\widetilde{m}} \widetilde{a}_j s^j \qquad \text{for } \operatorname{Re}(s) > \lambda. \tag{4.3}$$

By eliminating the common factors (if any), we can assume that the polynomials $\widetilde{P}(s)$ and $Q(s)$ are mutually prime.

Here λ is just any real number that is larger than the real part of all roots of Q. By Theorem 3.4, we can antitransform the function G in the framework of distributions:

$$G(s) = \sum_{k=0}^{m-n} c_k s^k \quad \text{for } \operatorname{Re}(s) > \lambda \quad \Rightarrow \quad \mathcal{L}^{-1}(G) = \sum_{k=0}^{m-n} c_k D^k \delta_0. \tag{4.4}$$

We are left with the calculus of the antitransform of the *proper* rational function $R(s) = \widetilde{P}(s)/Q(s)$, which is also of the form (4.1) but with $\widetilde{m} < n$.

Let $\{z_h : h = 1, ..., \ell\}$ be the distinct complex roots of the polynomial $Q(s)$, each of multiplicity r_h; thus $r_1 + ... + r_\ell = n$. As $Q(s) = b_n \prod_{h=1}^{\ell} (s - z_h)^{r_h}$, the function

$$R(s) = \frac{\widetilde{P}(s)}{b_n \prod_{h=1}^{\ell} (s - z_h)^{r_h}} \tag{4.5}$$

is defined for any $s \in \mathbb{C} \setminus \{z_h : h = 1, ..., \ell\}$. Thus $R(s)$ is defined whenever $\mathrm{Re}(s) > \max\{\mathrm{Re}(z_h) : h = 1, ..., \ell\}$. This function can be decomposed into a sum of *simple fractions:*

$$R(s) = \sum_{h=1}^{\ell} \sum_{k=1}^{r_h} \frac{\mu_{hk}}{(s - z_h)^k}\,, \tag{4.6}$$

via what is called a *partial fraction expansion.* The family of complex coefficients $\{\mu_{hk}\}$ can be identified by rewriting each summand of the right member as a sum of fractions, all with the same denominator $\prod_{h=1}^{\ell}(s - z_h)^{r_h}$.

For instance, let $(1 + s + is^2)/(s - 1)^3$ be one of the summands of $R(s)$. By setting

$$\frac{1 + s + is^2}{(s-1)^3} = \frac{a}{s-1} + \frac{b}{(s-1)^2} + \frac{c}{(s-1)^3} \tag{4.7}$$

with $a, b, c \in \mathbb{C}$ to be determined, we get

$$\frac{1 + s + is^2}{(s-1)^3} = \frac{a(s-1)^2 + b(s-1) + c}{(s-1)^3} = \frac{as^2 - 2as + a + bs - b + c}{(s-1)^3}\,, \tag{4.8}$$

By equating the numerators, we then get $a = i$, $b = 1 + 2i$, $c = 2 + i$.

By (1.21) and (1.26),

$$\mathcal{L}\Big(\frac{t^{k-1}}{(k-1)!} e^{zt} H(t)\Big) = \frac{1}{(s-z)^k} \qquad \forall z \in \mathbb{C}, \forall k \in \mathbb{N} \setminus \{0\}; \tag{4.9}$$

by antitransforming (4.6) we then get

$$[\mathcal{L}^{-1}(R)](t) = \sum_{h=1}^{\ell} \sum_{k=1}^{r_h} \mu_{hk} \frac{t^{k-1}}{(k-1)!} e^{z_h t} H(t) \qquad \forall t > 0. \tag{4.10}$$

We have thus proved the following result.

Theorem 4.1 *(Heaviside) (i) By partial fraction expansion, the quotient of any pair of complex polynomials $P(s)$ and $Q(s)$ can be rewritten in the form*

$$\frac{P(s)}{Q(s)} = \sum_{k=0}^{m-n} c_k s^k + \sum_{h=1}^{\ell} \sum_{k=1}^{r_h} \frac{\mu_{hk}}{(s - z_h)^k} \qquad \text{for } \mathrm{Re}(s) > \max_{h=1,\ldots,\ell} \mathrm{Re}(z_h), \tag{4.11}$$

for suitable coefficients $c_k, \mu_{hk} \in \mathbb{C}$. Here $z_1, ..., z_\ell$ are the distinct complex roots of the polynomial $Q(s)$, r_h is the multiplicity of z_h.

(ii) This function the Laplace transform of the distribution

$$\sum_{k=0}^{m-n} c_k D^k \delta_0 + \sum_{h=1}^{\ell} \sum_{k=1}^{r_h} \mu_{hk} \frac{t^{k-1}}{(k-1)!} e^{z_h t} H(t) \qquad \forall t > 0. \tag{4.12}$$

(iii) T is a regular distribution (i.e., a function) iff the degree of $P(s)$ is strictly smaller than that of $Q(s)$.

Remarks 4.2 (i) Let us set

$$g_h(t) = \sum_{k=1}^{r_h} \frac{t^{k-1}}{(k-1)!} e^{z_h t} H(t) \qquad \text{for } h = 1, ..., \ell.$$

For each h, then:
(1) If $\mathrm{Re}(z_h) < 0$, then $|g_h(t)|$ decays exponentially as $t \to +\infty$ (g_h is then called a *transient).*
(2) If $\mathrm{Re}(z_h) > 0$, then $|g_h(t)|$ grows exponentially as $t \to +\infty$.
(3) If $\mathrm{Re}(z_h) = 0$ and if the root is simple (i.e., $r_h = 1$), then g_h is bounded.
(4) If $\mathrm{Re}(z_h) = 0$ and if the root is multiple (i.e., $r_h > 1$), then g_h is unbounded.
(ii) Instead of using the partial fraction expansion and the transform tables as we did so far, one can represent $\mathcal{L}^{-1}(R)$ via the Riemann-Fourier formula (2.8), and then proceed to integration in the complex plane, see e.g. [Gilardi pp. 444-452].

* **Heaviside formula for simple roots.** Let us assume that all the roots z_h of the polynomial $Q(s)$ are simple, i.e., $Q(s) = b_n \prod_{h=1}^{n} (s - z_h)$, so that the expansion in simple fractions of R reads

$$R(s) = \frac{\widetilde{P}(s)}{Q(s)} = \sum_{h=1}^{n} \frac{\mu_h}{s - z_h}\,, \tag{4.13}$$

whence

$$[\mathcal{L}^{-1}(R)](t) = \sum_{h=1}^{n} \mu_h e^{z_h t} H(t) \qquad \forall t > 0. \tag{4.14}$$

Next we determine the complex coefficients $\mu_1, ..., \mu_n$ in terms of the polynomials $P(s)$ and $Q(s)$. For $k = 1, ..., n$, as $Q(z_k) = 0$ we have

$$\mu_k + \sum_{h \neq k} \frac{\mu_h}{s - z_h}(s - z_k) \overset{(4.13)}{=} \frac{\widetilde{P}(s)}{Q(s)}(s - z_k) = \frac{\widetilde{P}(s)}{[Q(s) - Q(z_k)]/(s - z_k)} \qquad \text{for } k = 1, ..., n. \tag{4.15}$$

By passing to the limit as $s \to z_k$, we get

$$\mu_k = \lim_{s \to z_k} \frac{\widetilde{P}(s)}{Q(s)}(s - z_k) = \frac{\widetilde{P}(z_k)}{Q'(z_k)} \qquad \text{for } k = 1, ..., n. \tag{4.16}$$

(4.14) thus yields the following classical result.

Theorem 4.3 *(Heaviside formula) Let $\widetilde{P}(s)$ and $Q(s)$ be two polynomials, with the degree of $\widetilde{P}(s)$ smaller than that of $Q(s)$. If all the roots z_h of Q are simple, then*

$$[\mathcal{L}^{-1}(\widetilde{P}/Q)](t) = \sum_{h=1}^{n} \frac{\widetilde{P}(z_h)}{Q'(z_h)} e^{z_h t} H(t) \qquad \forall t > 0. \tag{4.17}$$

* **Case of multiple roots.** For instance, let us consider the case of a single root of multiplicity $r \geq 1$; that is, $Q(s) = b(s - z)^r$ for some $b, z \in \mathbb{C}$. Let us look for an expansion in simple fractions of the form

$$R(s) = \frac{\widetilde{P}(s)}{Q(s)} = \frac{\widetilde{P}(s)}{b(s - z)^r} = \sum_{h=1}^{r} \frac{\mu_h}{(s - z)^h}. \tag{4.18}$$

For $k = 1, ..., r$ we have

$$\Big[D_s^{r-k}\Big\{(s-z)^r R(s)\Big\}\Big]_{s=z} = \Big[D_s^{r-k}\sum_{h=1}^{r}\mu_h(s-z)^{r-h}\Big]_{s=z} = (r-k)!\mu_k \qquad \text{for } k = 1, ..., r. \quad (4.19)$$

We have thus identified the coefficients of (4.18):

$$\mu_k = \frac{1}{(r-k)!}[D_s^{r-k}(s-z)^r R(s)]_{s=z} \qquad \text{for } k = 1, ..., r. \quad (4.20)$$

This is obviously extended if the polynomial $Q(s)$ has several multiple roots, by decomposing $R(s)$ as in (4.6).

Remark 4.4 So far we considered ratios of polynomials with complex coefficients, we decomposed them in simple fractions as in (4.11). and then antitransformed them by using the rules (1.17), (1.21) and (1.26), see (4.12). Polynomials of a complex variable with *real* coefficients can also be decomposed in simple fractions as

$$\frac{P(s)}{Q(s)} = \sum_{k=0}^{m-n} c_k s^k + \sum_{h=1}^{\ell}\sum_{k=1}^{r_h} \frac{\mu_{hk}}{(s-z_h)^k} + \sum_{h=1}^{m}\sum_{k=1}^{p_h} \frac{a_{hk}s + b_{hk}}{[(s-c_h)^2 + e_h^2]^k}, \quad (4.21)$$

with real coefficients $c_k, \mu_{hk}, z_h, a_{hk}, b_{hk}, c_h, e_h$ and with $e_h \neq 0$ for any h, k. (If $e_h = 0$ the corresponding fraction can be absorbed by the c_k and μ_{hk} terms.)

For instance, using also the rules (1.22) and (1.23), for $c \neq 0$)

$$\Big[\mathcal{L}^{-1}\Big(\frac{a(s-s_0)+b}{(s-s_0)^2+c^2}\Big)\Big](t) = \Big(\frac{b}{c}\sin(ct) + a\cos(ct)\Big)e^{ct}H(t). \qquad \square \quad (4.22)$$

Exercises

— Check that the function $\mathcal{L}^{-1}(R)$ is bounded if $\mathrm{Re}(z_h) < 0$ for any h, and that if $\mathcal{L}^{-1}(R)$ is bounded then $\mathrm{Re}(z_h) \leq 0$ for any h.

— Check that $u * D^n\delta_0 = D^n u$ for any $u \in L^1_{\mathrm{loc}}$ and any $n \in \mathbb{N}$.

5 Laplace Transform and Differential Equations

In this section we consider the initial-value problem for an ODE, for a second-order equation with constant complex coefficients. First we formulate this problem pointwise and solve it via the Laplace transform, then we reformulate it in the sense of distributions. and show the equivalence of these two approaches.

Formulation of the Cauchy problem in L^1_{loc}. Let us fix $\alpha, \beta, \gamma, y_0, y_1 \in \mathbb{C}$ ($\alpha \neq 0$), a function $f : \mathbb{R}^+ \to \mathbb{C}$, and let us search for a function $y : \mathbb{R}^+ \to \mathbb{C}$ such that

$$\begin{cases} P(d/dt)y := \alpha y'' + \beta y' + \gamma y = f(t) & \text{for } t > 0, \\ y(0) = y_0, \qquad y'(0) = y_1. \end{cases} \quad (5.1)$$

Let us assume that $f \in D_{\mathcal{L}}$ and search for a solution $y \in D_{\mathcal{L}}$. Assuming that such a solution actually exists and is Laplace-transformable, let us set $Y := \mathcal{L}(y)$, $F := \mathcal{L}(f)$ and transform both members of the equation. For the moment let us proceed informally postponing the determination of the half-plane of convergence of the transforms. A posteriori we shall then check that the determined solution has the assumed regularity.

Recalling that

$$\begin{aligned} \mathcal{L}(y') &= s\mathcal{L}(y) - y(0), \\ \mathcal{L}(y'') &= s\mathcal{L}(y') - y'(0) = s^2\mathcal{L}(y) - sy(0) - y'(0), \end{aligned} \tag{5.2}$$

we get the following algebraic equation in frequency:

$$\alpha\big[s^2Y(s) - sy_0 - y_1\big] + \beta\big[sY(s) - y_0\big] + \gamma Y(s) = F(s) \qquad \text{for } \operatorname{Re}(s) > \lambda(f). \tag{5.3}$$

Setting

$$P(s) := \alpha s^2 + \beta s + \gamma, \qquad \Phi(s) := \alpha y_0 s + \alpha y_1 + \beta y_0, \tag{5.4}$$

we rewrite this transformed equation in the form

$$P(s)Y(s) = F(s) + \Phi(s) \qquad \text{for } \operatorname{Re}(s) > \lambda(f). \tag{5.5}$$

$P(s)$ is called the *characteristic polynomial* of the differential operator $P(d/dt)$. Denoting by $s_1, s_2 \in \mathbb{C}$ its (possibly coincident) roots, we have

$$P(s) \neq 0 \qquad \text{for } \operatorname{Re}(s) > \max\{\operatorname{Re}(s_1), \operatorname{Re}(s_2)\}.$$

Note that restrictions on the convergence abscissa have no effect on the solution, since, by analytic continuation, the holomorphic function $Y(s)$ is uniquely determined by its restriction to any half-plane.

The equation (5.3) is thus equivalent to

$$Y(s) = \frac{F(s)}{P(s)} + \frac{\Phi(s)}{P(s)} \qquad \text{for } \operatorname{Re}(s) > \max\{\operatorname{Re}(s_1), \operatorname{Re}(s_2), \lambda(f)\} =: M. \tag{5.6}$$

By informally antitransforming both members of this equation, we get a solution of problem (5.1):

$$y = \mathcal{L}^{-1}\Big(\frac{F(s)}{P(s)}\Big) + \mathcal{L}^{-1}\Big(\frac{\Phi(s)}{P(s)}\Big) \qquad \text{in } \mathbb{R}^+. \tag{5.7}$$

The first summand depends on the source term f, and represents the *forced* (or *driven) response:* this is the output of the system in presence of vanishing initial data. The second summand depends on the initial data, and represents the *free response* of the system, namely the output of the system in absence of source term in the equation.

As by the Convolution Theorem 1.6

$$\mathcal{L}^{-1}\Big(\frac{F(s)}{P(s)}\Big) = \mathcal{L}^{-1}\Big(\frac{1}{P(s)}\Big) * \mathcal{L}^{-1}(F(s)) = \mathcal{L}^{-1}\Big(\frac{1}{P(s)}\Big) * f, \tag{5.8}$$

we have

$$y = \mathcal{L}^{-1}\Big(\frac{1}{P(s)}\Big) * f + \mathcal{L}^{-1}\Big(\frac{\Phi(s)}{P(s)}\Big) \qquad \text{in } \mathbb{R}^+. \tag{5.9}$$

As the functions $1/P(s)$ and $\Phi(s)/P(s)$ fulfill the assumptions of Corollary 2.9, $y \in D_{\mathcal{L}}$. It is easily checked that this is a solution of (5.1). The above procedure is thus justified.

Reformulation of the Cauchy problem in $\mathcal{D}'$. The Cauchy problem (5.1) can be written as a single equation in the sense of distributions.

If $y = y(t)$ is causal and absolutely continuous in $]0, +\infty[$ and if there exists $y(0+) \in \mathbb{C}$, then $y - y(0+)H$ is absolutely continuous in $\mathbb{R}$. Therefore, still denoting by Dy the derivative in the sense of distributions in $\mathbb{R}$ and by y' the almost everywhere derivative in $]0, +\infty[$,

$$D[y - y(0+)H(t)] = y' \qquad \text{i.e.,} \qquad Dy = y' + y(0+)\delta_0 \qquad \text{in } \mathcal{D}'. \tag{5.10}$$

Analogously, if y' is also absolutely continuous in $]0, +\infty[$ and if $y'(0+) \in \mathbb{C}$, then

$$D[y' - y'(0+)H(t)] = y'' \qquad \text{i.e.,} \qquad D(y') = y'' + y'(0+)\delta_0 \qquad \text{in } \mathcal{D}'. \tag{5.11}$$

Hence

$$D^2 y \overset{(5.10)}{=} D(y') + y(0+)D\delta_0 \overset{(5.11)}{=} y'' + y(0+)D\delta_0 + y'(0+)\delta_0 \qquad \text{in } \mathcal{D}'. \tag{5.12}$$

Defining the operator

$$P(D) := \alpha D^2 + \beta D + \gamma I \qquad \text{in } \mathcal{D}', \tag{5.13}$$

(5.10) and (5.12) yield the formula

$$P(D)y = \alpha y'' + \beta y' + \gamma y + \alpha y(0+)D\delta_0 + [\alpha y'(0+) + \beta y(0+)]\delta_0 \qquad \text{in } \mathcal{D}'. \tag{5.14}$$

This allows us to rewrite the Cauchy problem (5.1) as a single equation on the whole $\mathbb{R}$ in the sense of distributions for the causal function y:

$$P(D)y = f(t) + \alpha y^0 D\delta_0 + [\alpha y^1 + \beta y^0]\delta_0 \qquad \text{in } \mathcal{D}'. \tag{5.15}$$

Remark 5.1 A comparison with the solution of ODEs on the whole real line via Fourier transform is in order, see [Fourier chapter]. □

Use of the Laplace transform in $\mathcal{D}'$. The use of distributions allowed us to include the initial data into the forcing term, and this will yield a more synthetical approach to the Cauchy problem.

The second member of the equation (5.5) reads

$$G(s) := F(s) + \Phi(s) = F(s) + \alpha y_0 s + \alpha y_1 + \beta y_0 \qquad \text{for } \operatorname{Re}(s) > \lambda(f), \tag{5.16}$$

and that algebraic equation has the solution

$$Y(s) = \frac{G(s)}{P(s)} \qquad \text{for } \operatorname{Re}(s) > \max\{\operatorname{Re}(s_1), \operatorname{Re}(s_2), \lambda(f)\}. \tag{5.17}$$

As

$$\mathcal{L}^{-1}(\alpha y_0 s + \alpha y_1 + \beta y_0) = \alpha y_0 D\delta_0 + (\alpha y_1 + \beta y_0), \tag{5.18}$$

by antitransforming (5.17) and defining

$$h = \mathcal{L}^{-1}(1/P(s)), \tag{5.19}$$

we get a solution of problem (5.1) in the sense of distributions:

$$\begin{aligned} y &= \mathcal{L}^{-1}\Big(\frac{1}{P(s)}(F + \Phi)\Big) = \mathcal{L}^{-1}\Big(\frac{1}{P(s)}\Big) * [\mathcal{L}^{-1}(F) + \mathcal{L}^{-1}(\Phi)] \\ &= h * [f + \alpha y_0 D\delta_0 + (\alpha y_1 + \beta y_0)\delta_0] \\ &= h * f + \alpha y_0 Dh + (\alpha y_1 + \beta y_0)h \qquad \text{in } \mathbb{R}^+. \end{aligned} \tag{5.20}$$

We thus retrieved (5.9), and obtained a more explicit representation of the solution.

Comparison between Fourier and Laplace transforms for ODE. We already pointed out that the Fourier transform allows one to deal with ODEs with constant coefficients set on the whole $\mathbb{R}^N$ for any $N \geq 1$, whereas the Laplace transform is especially suited for initial-value problems in $\mathbb{R}$.

With the Fourier transform we searched for a solution in $\mathcal{S}'$, so we excluded exponential growth, whereas with the Laplace transform exponential growth is allowed.

With the Fourier transform we assumed the characteristic polynomial to have no imaginary roots, whereas with the Laplace transform these roots are allowed, and play an important role in the evaluation of the inversion of the transform.

The transfer function. [10] The approach of *linear system theory* can be applied to initial-value problems via the Laplace transform, analogously to what we saw for differential equations on the whole $\mathbb{R}$ via the Fourier transform. In this case the solution y linearly depends on the forcing term f and on the initial data y_0, y_1. This is more clearly seen by using the Laplace transform in $\mathcal{D}'$.

Let us first assume that

$$y_0 = y_1 = 0, \tag{5.21}$$

so that the system that maps the *input* f to the *output* u is linear (rather than affine). By using distributions this restriction will be lifted, as the initial data are included into the forcing term, see (5.15) and ahead.

In the terminology of system theory, the solution operator $L : f \mapsto y$ is called a *continuous filter.* Here we deal with the filter that is associated to problem (5.1); this analysis can however be extended to systems of first order equations, to higher order problems, and to a much more general set-up — including partial differential equations, integro-differential equations, and so on.

Let $h := L(\delta_0)$ be the *response in time to the unit impulse* δ_0*,* that is,

$$P(D)h = \delta_0 \qquad \text{in } \mathcal{D}'(\mathbb{R}), \tag{5.22}$$

or also, by (5.20),

$$h = \mathcal{L}^{-1}\Big(\frac{1}{P(s)}\Big) * \delta_0 = \mathcal{L}^{-1}\Big(\frac{1}{P(s)}\Big) \ (\in D_{\mathcal{L}}). \tag{5.23}$$

Note that $h \in D_{\mathcal{L}}$, because of the Antitransformability Theorem 2.8 (of Corollary 2.9 if $P(s)$ is or order one). This is called the *transfer function in time* of the system. By applying the transform $\mathcal{L}$, we get the *response in frequency to the unit impulse:*

$$[\mathcal{L}(h)](s) = [\mathcal{L}(L(\delta_0))](s) = \frac{1}{P(s)} \qquad \text{in } \mathbb{C}_{L(\delta_0)},$$

which is called the *transfer function* of the linear system in frequency, or the *spectrum* of the system.

Remark 5.2 Because of (5.22), the function h is what is called a *fundamental solution* of the differential equation $P(D)u = f$. This fundamental solution is not unique, as it is determined up to the sum of solutions of the homogeneous equation $P(D)v = 0$. But the injectivity of $\widehat{\mathcal{L}}$ entails that the function $h \in D_{\widehat{\mathcal{L}}}$ is the unique *causal* fundamental solution of the Cauchy problem (5.1) for $f = \delta_0$ and $y_1 = y_2 = 0$.

This is at variance with what we saw via the Fourier transform for the problem on the whole $\mathbb{R}$, cf. (**??**) [Fourier chapter]. □

[10] Here we anticipate some notions that we shall see in more detail in the chapter devoted to filters.

Two examples. For any $k > 0$ and any $f \in D_{\mathcal{L}}$, let us consider the differential equations

$$y_1'' - k^2 y_1 = f(t), \qquad y_2'' + k^2 y_2 = f(t); \tag{5.24}$$

the second one represents harmonic motion. These are respectively associated to the operators

$$P_1(D) := D^2 - k^2 I, \qquad P_2(D) := D^2 + k^2 I \qquad (I\text{: identity operator}),$$

with characteristic polynomials

$$P_1(s) := s^2 - k^2, \qquad P_2(s) := s^2 + k^2 \qquad (s \in \mathbb{C}),$$

and with roots

$$s_{11}, s_{12} = \pm k \quad \text{for } P_1(s), \qquad s_{21}, s_{22} = \pm ik \quad \text{for } P_2(s).$$

The general solution y_j $(j = 1, 2)$ of the corresponding Cauchy problem (5.1) has the form (5.9), with convergence abscissa

$$\lambda(y_1) \leq \max\{k, \lambda(f)\}, \qquad \lambda(y_2) \leq \max\{0, \lambda(f)\},$$

and transfer function in time

$$\begin{aligned} h_1(t) &= \mathcal{L}^{-1}\Big(\frac{1}{P_1(s)}\Big) = \frac{1}{k}\,\mathcal{L}^{-1}\Big(\frac{k}{s^2 - k^2}\Big) = \frac{\sinh kt}{k} H(t) \notin \mathcal{S}', \\ h_2(t) &= \mathcal{L}^{-1}\Big(\frac{1}{P_2(s)}\Big) = \frac{1}{k}\,\mathcal{L}^{-1}\Big(\frac{k}{s^2 + k^2}\Big) = \frac{\sin kt}{k} H(t) \in \mathcal{S}'. \end{aligned} \tag{5.25}$$

Note that, as $t \to +\infty$, $h_1(t)$ diverges exponentially whereas $h_2(t) \to 0$.

*** Higher-order equations.** Next we extend the previous analysis to initial-value problems for equations of any order $M \geq 1$.

Let $\alpha_0, ..., \alpha_M \in \mathbb{C}$ $(\alpha_M \neq 0)$ and define the differential operator $P(D) = \sum_{m=0}^{M} \alpha_m D^m$. Let us denote by $s_1, ..., s_M$ the (possibly repeated) complex roots of the characteristic polynomial $P(s) = \sum_{m=0}^{M} \alpha_m s^m$, and set $\widetilde{M} := \max\{\mathrm{Re}(s_i) : i = 1, ..., M\}$. For any $f : \mathbb{R}^+ \to \mathbb{C}$ and $y_0, ..., y_{M-1} \in \mathbb{C}$, let us search for $y : \mathbb{R}^+ \to \mathbb{C}$ such that

$$\begin{cases} \displaystyle\sum_{m=0}^{M} \alpha_m D^m y = f(t) & \text{for } t > 0 \\ D^m y(0) = y_m^0 & \text{for } m = 0, ..., M-1. \end{cases} \tag{5.26}$$

Let us assume that $f, y \in D_{\mathcal{L}}$, set $Y := \mathcal{L}(y)$, $F := \mathcal{L}(f)$, and apply the Laplace transform to (5.26). By the differentiation Theorem 2.2, (5.26) yields

$$\sum_{m=0}^{M} \alpha_m s^m Y(s) - \sum_{m=0}^{M} \alpha_m \sum_{n=0}^{m-1} s^{m-1-n} y_n^0 = F(s) \qquad \text{for } \mathrm{Re}(s) > \max\{\widetilde{M}, \lambda(f)\}. \tag{5.27}$$

(As we saw above for the second-order equation, here also we might reformulate the Cauchy problem as a single equation in the sense of distributions, getting of course the same transformed equation.) Setting

$$\Phi(s) := \sum_{m=0}^{M} \alpha_m \sum_{n=0}^{m-1} s^{m-1-n} y_n^0, \tag{5.28}$$

the equation (5.27) reads

$$P(s)Y(s) = F(s) + \Phi(s) \qquad \text{for } \operatorname{Re}(s) > \lambda(f). \tag{5.29}$$

As $P(s) \neq 0$ for $\operatorname{Re}(s) > \widetilde{M}$, we can divide both members by $P(s)$, getting

$$Y(s) = \frac{F(s)}{P(s)} + \frac{\Phi(s)}{P(s)} \qquad \text{for } \operatorname{Re}(s) > \max\{\widetilde{M}, \lambda(f)\}.$$

Defining the transfer function in time $h := \mathcal{L}^{-1}(1/P(s))$, the Convolution Theorem 1.6, (3.19) and $(5.28)_2$ yield

$$\begin{aligned} y &= \mathcal{L}^{-1}\Big(\frac{1}{P(s)}F\Big) + \mathcal{L}^{-1}\Big(\frac{1}{P(s)}\Phi\Big) = h * \mathcal{L}^{-1}(F) + h * \mathcal{L}^{-1}(\Phi) \\ &= h * f + h * \sum_{m=0}^{M} \alpha_m \sum_{n=0}^{m-1} y_n^0 \, D^{m-1-n}\delta_0 \\ &= h * f + \sum_{m=0}^{M} \alpha_m \sum_{n=0}^{m-1} y_n^0 \, D^{m-1-n} h \qquad \text{in } \mathbb{R}^+. \end{aligned} \tag{5.30}$$

These two summands are the free and the forced response. The function y has convergence abscissa $\lambda(y) \le \max\{\widetilde{M}, \lambda(f)\}$, and this justifies the previous transformations.

This discussion can be extended in several directions: e.g., to Cauchy problem for systems of differential equations, to equations and systems of differential equations with delay, and so on.

*** Linear systems of first-order equations.** Let

$$A \in \mathbb{C}^{N\times N} \ \ (N \ge 1), \quad f \in L^1_{\text{loc}}(\mathbb{R}^+;\mathbb{C}^N), \quad u^0 \in \mathbb{C}^N,$$

and consider the following first-order vector Cauchy problem, for the unknown function $u \in L^1_{\text{loc}}(\mathbb{R}^+;\mathbb{C}^N)$:

$$\begin{cases} D_t u = A \cdot u + f & \text{in } \mathbb{R}^+, \\ u(0) = u^0. \end{cases} \tag{5.31}$$

By extending u and f to causal functions $\mathbb{R} \to \mathbb{C}^N$, this problem can be formulated as the single vector equation

$$D_t u = A \cdot u + f + u^0 \delta_0 \qquad \text{in } \mathcal{D}'(\mathbb{R})^N. \tag{5.32}$$

Let us assume that f and u are both Laplace-transformable (componentwise), and set $U = \mathcal{L}(u)$, $F = \mathcal{L}(f)$. By transforming the equation (5.32), we get

$$(sI - A) \cdot U(s) = F(s) + u^0 \qquad \text{for } \operatorname{Re}(s) \text{ large enough}, \tag{5.33}$$

that is, defining the *resolvent matrix* $R(s) = (sI - A)^{-1}$,

$$U(s) = R(s) \cdot [F(s) + u^0] \qquad \text{for } \operatorname{Re}(s) \text{ large enough}. \tag{5.34}$$

More precisely, defining the (possibly nondistinct) complex eigenvalues $s_1, ..., s_N$ of the matrix A, (5.34) holds for $\operatorname{Re}(s) > \max_{j=1,...,N}\{\lambda(u_j), \lambda(f_j), \operatorname{Re}(s_j)\}$.

Let us now introduce the (causal) antitransformed matrix-function $S = \mathcal{L}^{-1}(R) : \mathbb{R} \to \mathbb{C}^{N\times N}$, namely, the solution of the problem

$$(sI - A)^{-1} = [\mathcal{L}(S)](s) \qquad \text{for } \mathrm{Re}(s) > \max_{j=1,\dots,N} \{\mathrm{Re}(s_j)\}. \tag{5.35}$$

By antitransforming the equation (5.34), we get

$$u = \mathcal{L}^{-1}(R) * \mathcal{L}^{-1}(F + u^0) = S * f + S * (u^0 \delta_0). \tag{5.36}$$

We thus retrieve the classical *formula of variation of parameters:*

$$u(t) = \int_0^t S(t-\tau) \cdot f(\tau)\, d\tau + S(t) \cdot u^0 \qquad \forall t > 0. \tag{5.37}$$

The function S represents the *semigroup* that is generated by the matrix A. [11] The equation (5.35) can then be interpreted as follows

$$\begin{array}{l}\text{the resolvent } (sI - A)^{-1} \text{ of the semigroup } S(t) \\ \text{coincides with the Laplace transform of that semigroup.}\end{array} \tag{5.38}$$

Notice that, by the definition of the exponential of a matrix,

$$S(t) = e^{At} H(t) = \sum_{n=0}^{\infty} \frac{A^n t^n}{n!} H(t) \ (\in \mathbb{C}^{N\times N}) \qquad \forall t \in \mathbb{R}. \tag{5.39}$$

*** Linear systems of higher-order equations.** Let A, f be as above, $m \in \mathbb{N}$, $a_n, u^n \in \mathbb{C}$ for $n = 0, ..., m-1$ ($a_m \neq 0$), and consider the following vector Cauchy problem of order m:

$$\begin{cases} \sum_{n=0}^{m} a_n D^n u(t) = A \cdot u(t) + f(t) & \text{for } t > 0, \\ D^n u(0) = u_n^0 & \text{for } n = 0, ..., m-1. \end{cases} \tag{5.40}$$

By extending u and f to causal functions $\mathbb{R} \to \mathbb{C}^N$, this problem can be formulated as a single vector equation

$$\sum_{n=0}^{m} a_n D^n u(t) = A \cdot u(t) + f(t) + \sum_{n=0}^{m-1} a_n u_n^0 D^{m-1-n} \delta_0 \qquad \text{in } \mathcal{D}'(\mathbb{R})^N. \tag{5.41}$$

Let us assume that $f, y \in D_{\mathcal{L}}$, set $U = \mathcal{L}(u)$, $F = \mathcal{L}(f)$, and define the characteristic polynomial $P(s)$ and the polynomial $\Phi(s)$:

$$P(s) = \sum_{n=0}^{m} a_n s^n, \qquad \Phi(s) = \sum_{n=0}^{m-1} a_n u_n^0 s^{m-1-n} \quad \forall s \in \mathbb{C}. \tag{5.42}$$

By transforming the equation (5.41), we get

$$[P(s)I - A] \cdot U(s) = F(s) + \Phi(s) \qquad \text{for } \mathrm{Re}(s) > \max\{\lambda(u), \lambda(f)\}, \tag{5.43}$$

[11] A semigroup is a family of operators $S : [0, +\infty[\to \mathcal{L}(X)$ (the space of linear and continuous operators on a Banach space X) such that $S(0) = I$, $S(t_1 + t_2) = S(t_1) \circ S(t_2)$ for any $t_1, t_2 \geq 0$, and $t \mapsto S(t)$ is continuous.

that is, defining the matrix function $R(s) = [P(s)I - A]^{-1}$,

$$U(s) = R(s) \cdot [F(s) + \Phi(s)] \qquad \text{for Re}(s) \text{ large enough.} \tag{5.44}$$

Let us introduce the matrix antitransform $S = \mathcal{L}^{-1}(R) : \mathbb{R} \to \mathbb{C}^{N \times N}$, that is, the solution of the problem

$$[P(s)I - A]^{-1} = [\mathcal{L}(S)](s) \qquad \text{for Re}(s) \text{ large enough.} \tag{5.45}$$

By antitransforming (5.44) we get

$$u = S * f + S * \mathcal{L}^{-1}(\Phi) = S * f + S * \sum_{n=1}^{m} \sum_{j=0}^{n-1} a_n u_j D^{n-h-1} h \qquad \text{in } \mathbb{R}. \tag{5.46}$$

Remark 5.3 This approach can be extended to linear evolutionary PDEs, e.g. of the form $D_t u + Au = f$. In this case the associated stationary operator A is defined in an infinite-dimensional space (typically a Sobolev space). □

6 Harmonic Oscillator

Harmonic oscillator. This model is ubiquitous in applied sciences. For instance, it may represent finite-dimensional mechanical systems (e.g., a pendulum, a swing, a system of masses and springs), continuous materials (so-called rheological models), acoustic models (e.g., the vibrator of a musical instrument), electric circuits (e.g., an inductance and a capacitance in series), and several other phenomena or devices.

For instance, for a system having unit mass and *characteristic* (or *natural) frequency* $\omega > 0$ submitted to a forcing term $f \in D_{\mathcal{L}}$, the equation of (undamped) harmonic motion reads

$$y'' + \omega^2 y = f(t) \qquad \text{for } t > 0. \tag{6.1}$$

We shall first derive the solutions of this equation via Laplace transform, and then include a first-order damping term. In particular we shall discuss the effect of a periodic forcing.

By applying the Laplace transform to (6.1), we get

$$(s^2 + \omega^2)\widetilde{y} = sy(0) + y'(0) + \widetilde{f}(s) \qquad \text{for Re}(s) \text{ large enough.} \tag{6.2}$$

Dividing both members by the characteristic polynomial $s^2 + \omega^2$ and applying the rules (1.22) and (1.23), by (5.7) we get

$$y(t) = y(0)\cos(\omega t) + y'(0)\sin(\omega t) + \frac{1}{\omega}\int_0^t f(t-\tau)\sin(\omega\tau)\, d\tau \qquad \forall t > 0. \tag{6.3}$$

The first two summands are the *free response* of the system, and also read

$$y_1(t) := y(0)\cos(\omega t) + y'(0)\sin(\omega t) = A\cos(\omega t - \phi) \qquad \forall t > 0, \tag{6.4}$$

for suitable constants $A \geq 0$ and ϕ. The integral term of (6.3), which we denote by $y_2(t)$, is the *forced* or *(driven) response.*

Let us now assume that the forcing term is an oscillating function $f(t) = A_0 \cos(\omega_0 t)$, with $A_0 \geq 0$ and $\omega_0 \neq \omega$. Note that

$$\begin{aligned} \widetilde{y}_2(t) &= \frac{\widetilde{f}(s)}{s^2+\omega^2} = \frac{A_0 s}{s^2+\omega_0^2}\,\frac{1}{s^2+\omega^2} \\ &= \frac{A_0}{\omega_0^2-\omega^2}\Big(\frac{s}{s^2+\omega^2} - \frac{s}{s^2+\omega_0^2}\Big) \qquad \forall t>0, \end{aligned} \tag{6.5}$$

and by antitransforming we get

$$y_2(t) = \frac{1}{\omega}\int_0^t f(t-\tau)\sin(\omega\tau)\,d\tau = \frac{A_0}{\omega_0^2-\omega^2}[\cos(\omega t) - \cos(\omega_0 t)] \qquad \forall t>0. \tag{6.6}$$

This represents the sum of two oscillations, which respectively have the characteristic frequency ω of the oscillator and the forcing frequency ω_0, On the other hand, they have the same amplitude $A_0/(\omega_0^2 - \omega^2)$, and this depends on both frequencies.

As $\omega_0 \to \omega$, by the Hôpital rule $y_2(t)$ converges to

$$y_2(t)|_{\omega_0=\omega} = A_0 \frac{\frac{d}{d\omega_0}[\cos(\omega t) - \cos(\omega_0 t)]}{\frac{d}{d\omega_0}[\omega_0^2-\omega^2]}\Bigg|_{\omega_0=\omega} = \frac{A_0 t}{2\omega}\sin(\omega t) \qquad \forall t>0. \tag{6.7}$$

The amplitude of this function diverges as $t \to +\infty$. Physically this corresponds to the phenomenon of *resonance,* which indeed occurs in an undamped harmonic oscillator whenever the forcing frequency coincides with the characteristic frequency.

*** Damped harmonic oscillator.** Let us next assume that the oscillator is damped by a dissipative force, e.g. the resistance of the medium for a mechanical oscillator, a friction element in a rheological model, a resistance *in series* with inductance and capacitance in an electric circuit. In this case the equation of motion has the form

$$y'' + 2ky' + \omega^2 y = f(t) \qquad \text{for } t>0, \tag{6.8}$$

for a constant $k > 0$. By applying the Laplace transform and dividing by the characteristic polynomial, we get

$$\widetilde{y} = \frac{(s+2k)y(0) + y'(0)}{s^2+2ks+\omega^2} + \frac{\widetilde{f}(s)}{s^2+2ks+\omega^2} \qquad \text{for } \mathrm{Re}(s) \text{ large enough.} \tag{6.9}$$

Let us set $\sigma = \sqrt{\omega^2 - k^2}$ and first assume that $\sigma \neq 0$. [12] As

$$\frac{1}{s^2+2ks+\omega^2} = \frac{1}{(s+k)^2+\sigma^2} \overset{(4.22)}{=} \frac{1}{\sigma}\mathcal{L}(e^{-kt}\sin(\sigma t)H(t)), \tag{6.10}$$

$$\frac{s+k}{s^2+2ks+\omega^2} = \frac{s+k}{(s+k)^2+\sigma^2} \overset{(4.22)}{=} \mathcal{L}(e^{-kt}\cos(\sigma t)H(t)), \tag{6.11}$$

by antitransforming (6.9) we get

$$\begin{aligned} &y(t) = y_1(t) + y_2(t) \qquad \forall t>0, \\ &y_1(t) = y(0)e^{-kt}\cos(\sigma t) + \frac{1}{\sigma}[ky(0)+y'(0)]e^{-kt}\sin(\sigma t) \qquad \text{(free response)}, \\ &y_2(t) = \frac{1}{\sigma}\int_0^t f(t-\tau)e^{-k\tau}\sin(\omega\tau)\,d\tau \qquad \text{(forced response)}. \end{aligned} \tag{6.12}$$

[12] Here we are not assuming $\omega^2 > k^2$.

Here we study the free response $y_1(t)$ distinguishing three cases:

(i) If $0 < k < \omega$ *(underdamping)* then $\sigma^2 > 0$, so that σ is real. The free response represents *damped oscillations,* and can be rewritten in the form

$$y_1(t) = Ce^{-kt}\cos(\sigma t + \theta) \qquad \forall t > 0, \tag{6.13}$$

for suitable constants $C \geq 0$ and θ. Note that $y_1(t) \to 0$ as $t \to +\infty$.

(ii) If $k > \omega$ *(overdamping)* then $\sigma^2 < 0$, so that σ is imaginary. Replacing the trigonometric functions by hyperbolic functions, (6.12) thus reads

$$\begin{aligned} y(t) &= y_1(t) + y_2(t) \qquad \forall t > 0, \\ y_1(t) &= y(0)e^{-kt}\cosh(|\sigma|t) + \frac{1}{|\sigma|}[ky(0) + y'(0)]e^{-kt}\sinh(|\sigma|t), \\ y_2(t) &= \frac{1}{|\sigma|}\int_0^t f(t-\tau)e^{-k\tau}\sinh(|\sigma|\tau)\,d\tau, \end{aligned} \tag{6.14}$$

and these functions do not oscillate. As $t \to +\infty$, the hyperbolic functions diverge, but the factor $e^{-k\tau}$ prevails since $k > |\sigma|$. Therefore $y_1(t) \to 0$ as $t \to +\infty$ also in this case.

(iii) If $\omega = k$ *(critical damping)* then $\sigma = 0$. As

$$\frac{1}{s^2 + 2ks + k^2} = \frac{1}{(s+k)^2} = \mathcal{L}(te^{-kt}H(t)), \tag{6.15}$$

$$\frac{s+k}{s^2 + 2ks + k^2} = \frac{1}{s+k} = \mathcal{L}(e^{-kt}H(t)), \tag{6.16}$$

in this case we get

$$\begin{aligned} y(t) &= y_1(t) + y_2(t) \qquad \forall t > 0, \\ y_1(t) &= y(0)e^{-kt} + [y'(0) + ky(0)]te^{-kt}, \\ y_2(t) &= \int_0^t f(t-\tau)e^{-k\tau}\,d\tau, \end{aligned} \tag{6.17}$$

(This equality may also be derived by passing to the limit as $\omega \to k$ in (6.13) and (6.14).) So here we have no oscillation, and again $y_1(t) \to 0$ as $t \to +\infty$.

In each of these three cases $y_1(t)$ is a *transient* (i.e., it vanishes as $t \to +\infty$). This is quite natural, since the free response is influenced by damping, and not by the energy that the forcing term may provide to the system.

Remarks 6.1 (i) The free response $y_1(t)$ can also be derived via a popular rule, which surrogates the Laplace transform. This consists in determining the roots of the characteristic polynomial, here $-k \pm i\sigma$, and then representing $y_1(t)$ as

$$y_1(t) = C_1 e^{-(k+i\sigma)t} + C_2 e^{-(k-i\sigma)t} \qquad \forall t > 0, \text{ if } \sigma \neq 0, \tag{6.18}$$

$$y_1(t) = C_1 e^{-kt} + C_2 te^{-kt} \qquad \forall t > 0, \text{ if } \sigma = 0. \tag{6.19}$$

Expressing the constants C_1 and C_2 in terms of $y(0)$ and $y'(0)$, one retrieves $y_1(t)$ as in (6.13), (6.14) and (6.17), respectively for $\sigma^2 > 0$, $\sigma^2 < 0$ and $\sigma = 0$.

(ii) For $\sigma \neq 0$ the representations (6.13) and (6.14) are equivalent, although the first one is real for $\sigma > 0$ and the second one for $\sigma < 0$. [13] □

[13] As a rule, a real representation, when available, is usually preferred to a complex one, although the acute title of a recent book reads *imaginary numbers are real.*

Finally, we come to damped evolution in presence of an oscillatory forcing term, e.g. $f(t) = A_0 \cos(\omega_0 t)$. We just state a couple of results, and refer e.g. to [Debnath-Bhatta, Sect. 4.2] for details.

In this case the transformed equation (6.9) reads

$$(s^2 + 2ks + \omega^2)\widetilde{y} = \frac{A_0 s}{s^2 + \omega_0^2} + (s + 2k)y(0) + y'(0) \qquad \text{for } \mathrm{Re}(s) \text{ large enough.} \tag{6.20}$$

For $k < \omega_0$, one can show that its solution is of the form

$$y(t) = A_1 e^{-kt} \cos(\sigma t - \phi_1) + A_2 \cos(\omega_0 t - \phi_2) =: y_1(t) + y_2(t) \qquad \forall t > 0, \tag{6.21}$$

wherein A_1, A_2, ϕ_1, ϕ_2 are constants, and $\sigma = \sqrt{\omega^2 - k^2}$ as above. More specifically, $A_1(\geq 0)$ and ϕ_1 depend on all the parameters k, ω, ω_0, $y(0)$, $y'(0)$. On the other hand, $A_2(\geq 0)$ and ϕ_2 are determined by k, ω, ω_0, A_0, and are independent of $y(0), y'(0)$. As one might expect, as $k \to 0$ we retrieve (6.6).

The function $y_1(t)$ is a transient response: it represents a damped oscillation, which decays to 0 as $t \to +\infty$. The solution $y(t)$ is thus asymptotic to $y_2(t)$, which represents a steady oscillation and has the frequency of the forcing term. As $\omega_0 \to \omega$ and $k \to 0$ the amplitude A_2 diverges, and so one retrieves the phenomenon of resonance. For $k > 0$, large but finite values of this amplitude appear also for ω_0 close to ω.