Partial Differential Equations – A. Visintin – 2016

Outline: The modern theory of PDEs is based upon several analytical tools. In particular it has been and is still developed in function spaces, and thus rests upon functional analysis.

This course will illustrate the main properties of linear PDEs of first and second order, will introduce the Sobolev spaces, and will illustrate the weak formulation of boundary- and initial-value problems for second-order PDEs.

Prerequisites (in parentheses the corresponding courses in Trento):

Differential and Integral Calculus, with Fourier series and ODEs *(Analisi I, II e III).*

Measure theory and Lebesgue integration *(Analisi III).*

Linear algebra *(Geometria I).*

General topology *(Geometria II).*

Elementary notions on PDEs *(Fisica-Matematica).*

Banach and Hilbert spaces, functional spaces, linear and continuous operators *(Analisi Funzionale).*

Fourier and Laplace transforms *(Fourier Analysis,* at the first semester).

The parallel course *Advanced Analysis* is recommended.

Introduction to the linear theory of partial differential equations. The main classes of second-order equations will be addressed, keeping an eye on mathematical-physical applications. The Sobolev spaces will be introduced, and used for the weak theory of initial- and boundary-value problems.

Main Issues:

1. Basic Linear Second-Order PDEs [Renardy-Rogers; chap. 1]

Review of ordinary differential equations (ODEs). Existence and uniqueness for initial-value problems. Eigenvalue problem for the homogeneous boundary-value problem for equations like $y'' + \lambda y = 0$.

Linear systems of ODEs. Fundamental solution and matrix function. Formula of variation of parameters. Gronwall's lemma.

Laplace/Poisson equation: boundary conditions. Solution by separation of variables.

Green identities. Variational formulation of the Laplace/Poisson equation. Fundamental lemma of the calculus of variations. Maximum principle.

Heat equation: boundary and initial conditions. Solution by separation of variables. Backward heat equation. Energy inequality.

Duhamel principle for evolutionary PDEs. Variational formulation. Energy inequality. Maximum principle.

Wave equation: boundary and initial conditions. Solution by separation of variables. D'Alembert solution of the wave equation. Domain of dependence and domain of influence. Variational formulation. Energy conservation. Schrödinger equation.

Comparison between the qualitative properties of the heat, wave and Schrödinger equations.

Linear elliptic operators in nondivergence and divergence forms.

Weak and strong maximum form of the maximum principle for linear elliptic and parabolic equations in nondivergence form in C^0. Monotone dependence of the solution on the data. Uniqueness of the solution. [Renardy-Rogers; chap. 4]

2. Characteristics and First-Order PDEs [Renardy-Rogers; chap. 2], [Lecture Notes]

Multi-indices. Principal part and symbol of a linear differential operator. Classification of nonlinear PDEs: quasilinear, semilinear, fully-nonlinear equations.

Classification of linear second-order PDEs: elliptic, parabolic, hyperbolic.

Characteristics of linear and quasilinear equations. Projected and unprojected characteristic curves of first-order equations.

Integration of linear and quasilinear first-order equations via the method of characteristics. Lagrange method of first integrals.

Statement of the Cauchy-Kovalevskaya and Holmgren theorems (without proofs).

3. Sobolev Spaces [Lecture Notes]

Spaces of Hölder class.

Euclidean domains of Hölder class. Cone property.

Sobolev spaces of positive order. Characterization of the dual.

Extension operators. Theorem of Caldéron-Stein. The method of extension by reflection.

Density results: interior approximation (Mayer-Serrin theorem) and exterior approximation.

Sobolev inequality and imbedding between Sobolev spaces. Morrey theorem. Sobolev and Morrey indices. Rellich-Kondrachov compactness theorem.

L^p- and Sobolev spaces on manifolds. Trace theorems.

Friedrichs inequality.

Sobolev spaces of real order and Bessel spaces.

4. Weak Formulation of Second-Order PDEs [Lecture Notes]

Elliptic operators. Classical, strong and weak formulation.

Operators in divergence or nondivergence form. Conormal derivative.

Strong and weak formulation of second-order elliptic equations in divergence form, coupled with boundary-conditions (i.e., either Dirichlet or Neumann or mixed or periodic conditions).

Strong and weak formulation of the Cauchy problem for second-order parabolic/hyperbolic equations in divergence form, coupled with initial- and boundary-conditions.

Procedure of approximation, a priori estimate and passage to the limit for elliptic, parabolic and hyperbolic equations. Role of symmetry for parabolic and hyperbolic problems.

Elliptic problems governed by a symmetric operator are equivalent to minimization problems.

Theorem of Lax-Milgram. Application to the analysis of PDEs in divergence form.

Spaces of functions of time ranging in Banach spaces. Use of these spaces in the formulation of initial and boundary-value problems for parabolic and hyperbolic equations.

Reference textbook of the course:

M. Renardy, R. Rogers: An introduction to partial differential equations. Springer-Verlag, New York, 2004

Teacher's lecture notes (partly already available on the web).

Complementary textbooks:

Yu.V. Egorov, M.A. Shubin: Foundations of the classical theory of partial differential equations. Springer, Berlin 1992

L.C. Evans: Partial differential equations. American Mathematical Society, Providence, RI, 2010

S. Salsa: Equazioni a derivate parziali: metodi, modelli e applicazioni. Springer Italia, Milano 2003 (English edition also available)

S. Salsa, G. Verzini: Equazioni a derivate parziali: complementi ed esercizi. Springer Italia, Milano 2005

S. Salsa et al.: A primer on PDEs. Springer Italia, Milano 2013. (This is a modified English translation of the above text of Salsa; it also includes the correction of several exercises.)

Modality of exam:

Written and oral examinations.