Notes on Sobolev Spaces — A. Visintin

Contents: 1. Distributions. 2. Regularity of Euclidean domains. 3. Sobolev spaces of positive integer order. 4. Sobolev spaces of real integer order and traces. 5. Sobolev and Morrey imbeddings.

Note. The bullet $\bullet$ and the asterisk $*$ are respectively used to indicate the most relevant results and complements. The symbol [] follows statements the proof of which has been omitted, whereas [Ex] is used to propose the reader to fill in the argument as an exercise.

Here are some abbreviations that are used throughout:

a.a. = almost any; resp. = respectively; w.r.t. = with respect to.

p': conjugate exponent of p, that is, $p' := p/(p-1)$ if $1 < p < +\infty$, $1' := \infty$, $\infty' := 1$.

$\mathbf{N}_0 := \mathbf{N} \setminus \{0\}$; $\mathbf{R}^N_+ := \mathbf{R}^{N-1} \times]0, +\infty[$. $|A| :=$ measure of the measurable set A.

1. Distributions: omissis

2. Regularity of Euclidean Domains

Several notions may be used to define the regularity of a Euclidean open set Ω, or rather that of its boundary Γ. Here we just introduce two of them.

Open Sets of Class $C^{m,\lambda}$. Let us denote by $B_N(x,R)$ the ball of $\mathbf{R}^N$ of center x and radius R. For any $m \in \mathbf{N}$ and $0 \le \lambda \le 1$, we say that Ω is of class $C^{m,\lambda}$ (here $C^{m,0}$ stays for C^m), and write $\Omega \in C^{m,\lambda}$, iff for any $x \in \Gamma$ there exist:

(i) two positive constants $R = R_x$ and δ,

(ii) a mapping $\varphi : B_{N-1}(x,R) \to \mathbf{R}$ of class $C^{m,\lambda}$,

(iii) a Cartesian system of coordinates $y_1, ..., y_N$,

such that the point x is characterized by $y_1 = ... = y_N = 0$ in this Cartesian system, and, for any $y' := (y_1, ..., y_{N-1}) \in B_{N-1}(x,R)$,

$$\begin{array}{lll} y_N = \varphi(y') & \Rightarrow & (y', y_N) \in \Gamma, \\ \varphi(y') < y_N < \varphi(y') + \delta & \Rightarrow & (y', y_N) \in \Omega, \\ \varphi(y') - \delta < y_N < \varphi(y') & \Rightarrow & (y', y_N) \notin \bar{\Omega}. \end{array} \tag{2.1}$$

This means that Γ is an $(N-1)$-dimensional manifold (without boundary) of class $C^{m,\lambda}$, and that Ω stays only on one side of Γ. We say that Ω is a continuous (Lipschitz, Hölder, resp.) open set whenever it is of class C^0 ($C^{0,1}$, $C^{0,\lambda}$ for some $\lambda \in]0,1]$, resp.). [1]

For instance, the domain

$$\Omega_{a,b,\lambda} := \{(x,y) \in \mathbf{R}^2 : x > 0, ax^{1/\lambda} < y < bx^{1/\lambda}\} \qquad \forall \lambda \le 1, \forall a,b \in \mathbf{R}, a < b \tag{2.2}$$

is of class $C^{0,\lambda}$ iff $a < 0 < b$. [Ex]

We say that Ω is **uniformly of class $C^{m,\lambda}$** iff

$$\Omega \in C^{m,\lambda}, \quad \inf_{x\in\Gamma} R_x > 0, \quad \sup_{x\in\Gamma} \|\varphi_x\|_{C^{m,\lambda}} < +\infty. \tag{2.3}$$

For instance this is fulfilled by any bounded domain Ω of class $C^{m,\lambda}$. [Ex]

Cone Property. The above notion of regularity of open sets is not completely satisfactory, in that it excludes natural sets like balls with deleted center. We then introduce a further regularity notion.

(1) This notation refers to the Hölder spaces, that are defined half-a-page below ...

We say that Ω has the cone property iff there exist $a, b > 0$ such that, defining the finite open cone

$$C_{a,b} := \{x := (x_1, ..., x_N) : x_1^2 + ... + x_{N-1}^2 \leq b x_N^2,\ 0 < x_N < a\},$$

any point of Ω is the vertex of a cone contained in Ω and congruent to $C_{a,b}$. For instance, any ball with deleted center and the plane domains

$$\begin{aligned} &\Omega_1 := \{(\rho, \theta) : 1 < \rho < 2, 0 < \theta < 2\pi\} \qquad (\rho, \theta : \text{polar coordinates}), \\ &\Omega_2 := \{(x, y) \in \mathbf{R}^2 : |x|, |y| < 1, x \neq 0\} \end{aligned} \tag{2.4}$$

have the cone property, but are not of class C^0. [Ex]

Proposition 2.1 *Any bounded Lipschitz domain has the cone property. [Ex]*

For unbounded Lipschitz domains this may fail; $\Omega := \{(x, y) \in \mathbf{R}^2 : x > 1, 0 < y < 1/x\}$ is a counterexample. Note that a domain Ω is bounded whenever it has the cone property and $|\Omega| < +\infty$. []

Hölder Spaces. Let us fix any $\lambda \in]0, 1]$. The bounded continuous functions $v : \Omega \to \mathbf{C}$ such that

$$\sup_{x,y \in \Omega, x \neq y} \frac{|v(x) - v(y)|}{|x - y|^\lambda} < +\infty$$

are said **Hölder-continuous** of exponent (or index) λ, and form a vector Banach space, that we denote by $C^{0,\lambda}(\bar{\Omega})$, equipped with the norm

$$\|v\| := \sup_{x \in \Omega} |v(x)| + \sup_{x,y \in \Omega, x \neq y} \frac{|v(x) - v(y)|}{|x - y|^\lambda}\ . \tag{2.5}$$

If $\lambda = 1$ these functions are said **Lipschitz-continuous.**

It is known that, for any $m \in \mathbf{N}$, the functions $\Omega \to \mathbf{C}$ that are bounded and uniformly continuous jointly with their derivatives up to order m form a Banach space denoted by $C^m(\bar{\Omega})$. The vector space of the functions $\Omega \to \mathbf{C}$ that are bounded with their derivatives up to order m, and whose derivatives are Hölder-continuous of exponent λ, form a Banach space equipped with the norm

$$\sum_{|\alpha| \leq m} \sup_{x \in \Omega} |D^\alpha v(x)| + \sum_{|\alpha| = m} \sup_{x,y \in \Omega, x \neq y} \frac{|D^\alpha v(x) - D^\alpha v(y)|}{|x - y|^\lambda}\ , \tag{2.6}$$

that we denote by $C^{m,\lambda}(\bar{\Omega})$.

Some Imbeddings. We say that a topological space A is **imbedded** in another topological space B whenever $A \subset B$ and the injection operator $A \to B$ (which is then called an *imbedding)* is continuous.

Setting $C^{m,0} := C^m$ for any $m \in \mathbf{N}$, we have the next obvious imbeddings:

$$\begin{aligned} &C^{m,\lambda}(\bar{\Omega}) \subset C^{m,\nu}(\bar{\Omega}) \qquad \forall \lambda, \nu \in]0, 1], \nu \leq \lambda, \\ &C^{m+1,\lambda}(\bar{\Omega}) \subset C^{m,\lambda}(\bar{\Omega}) \qquad \forall m \in \mathbf{N}, \forall \lambda \in [0, 1]. \end{aligned} \tag{2.7}$$

Proposition 2.2 *Let either $\Omega = \mathbf{R}^N$, or $\Omega = \mathbf{R}^{N-1} \times]0, +\infty[\ \big(=: \mathbf{R}^N_+\big)$, or $\Omega \in C^{0,1}$ and bounded. Then:*

(i) For any $m_1, m_2 \in \mathbf{N}$, for any $\lambda_1, \lambda_2 \in [0, 1]$,

$$C^{m_2,\lambda_2}(\bar{\Omega}) \subset C^{m_1,\lambda_1}(\bar{\Omega}) \qquad \textit{if either} \quad m_1 < m_2, \quad \textit{or} \quad m_1 = m_2, \lambda_1 \leq \lambda_2. \tag{2.8}$$

(ii) Any function of $C^{m,\lambda}(\bar{\Omega})$ can be extended to an element of $C^{m,\lambda}(\mathbf{R}^N)$. [On the other hand, the restriction obviously preserves the regularity.] []

The next counterexample shows that some regularity is actually needed for (2.8) to hold. Let us fix any $\lambda \in \,]0,1[$, and set

$$\Omega := \{(x,y) \in \mathbf{R}^2 : x^2 + y^2 < 1,\ y < |x|^{1/2}\}. \tag{2.9}$$

Of course $\Omega \in C^{0,1/2} \setminus C^{0,\nu}$ for any $\nu > 1/2$. For any $a \in \,]1,2[$, the function $v : \Omega \to \mathbf{R} : (x,y) \mapsto (y^+)^a \operatorname{sign}(x)$ belongs to $C^1(\bar{\Omega})$ ($=: C^{1,0}(\bar{\Omega})$), but not to $C^{0,\nu}(\bar{\Omega})$ for any $\nu > a/2$. [Ex]

The next statement provides a procedure to construct new normed spaces, and is easily extended from the product of two spaces to that of a finite family.

Proposition 2.3 *Let A and B be two normed spaces and $p \in [1,+\infty]$. Then:*
*(i) The vector space $A \times B$ is a normed space equipped with the p-*norm *of the product:*

$$\begin{aligned} &\|(v,w)\|_p := (\|v\|_A^p + \|w\|_B^p)^{1/p} \qquad \text{if } 1 \le p < +\infty, \\ &\|(v,w)\|_\infty := \max\{\|v\|_A, \|w\|_B\}. \end{aligned} \tag{2.10}$$

We denote this space by $(A \times B)_p$. These norms are mutually equivalent.
(ii) If A and B are Banach spaces, then $(A \times B)_p$ is a Banach space.
(iii) If A and B are separable (reflexive, resp.), then $(A \times B)_p$ is also separable (reflexive, resp.).
(iv) If A and B are uniformly convex and $1 < p < +\infty$, then $(A \times B)_p$ is uniformly convex.
(v) If A and B are inner-product spaces (Hilbert spaces, resp.), equipped with the scalar product $(\cdot,\cdot)_A$ and $(\cdot,\cdot)_B$, resp., then $(A \times B)_2$ is an inner-product space (a Hilbert space, resp.) equipped with the scalar product

$$\big((u_1,v_1),(u_2,v_2)\big)_2 := (u_1,u_2)_A + (v_1,v_2)_B \qquad \forall (u_1,v_1),(u_2,v_2) \in (A \times B)_2.$$

$\|(\cdot,\cdot)\|_2$ is then the corresponding Hilbert norm.
(vi) $F \in (A \times B)'_p$ (the dual space of $(A \times B)_p$) iff there exists a (unique) pair $(g,h) \in A' \times B'$ such that

$$\langle F,(u,v)\rangle = {}_{A'}\langle g,u\rangle_A + {}_{B'}\langle h,v\rangle_B \qquad \forall (u,v) \in (A \times B)_p. \tag{2.11}$$

In this case

$$\|F\|_{(A\times B)'_p} = \|(g,h)\|_{(A'\times B')_{p'}}. \tag{2.12}$$

The mapping $(A \times B)'_p \to (A' \times B')_{p'} : F \mapsto (g,h)$ is indeed an isometric surjective isomorphism.

3. Sobolev Spaces of Positive Integer Order

In this section we introduce the Sobolev spaces of positive integer order, which consist of the complex-valued functions defined on a domain $\Omega \subset \mathbf{R}^N$ that fulfill certain integrability properties jointly with their distributional derivatives. We then see how these functions can be extended to $\mathbf{R}^N$ preserving their Sobolev regularity, and approximate them by smooth functions.

Sobolev Spaces of Positive Integer Order. Henceforth we shall denote by D derivatives in the sense of distributions. For any domain Ω of $\mathbf{R}^N$, any $m \in \mathbf{N}$ and any $p \in [1,+\infty]$, we set

$$W^{m,p}(\Omega) := \big\{v \in L^p(\Omega) : D^\alpha v \in L^p(\Omega),\ \forall \alpha \in \mathbf{N}^N, |\alpha| \le m\big\}. \tag{3.1}$$

(Thus $W^{0,p}(\Omega) := L^p(\Omega)$.) This is a vector space over $\mathbf{C}$, that we equip with the norm

$$\|v\|_{W^{m,p}(\Omega)} := \Big(\sum_{|\alpha|\le m} \|D^\alpha v\|^p_{L^p(\Omega)} \Big)^{1/p} \qquad \forall p \in [1,+\infty[, \tag{3.2}$$

$$\|v\|_{W^{m,\infty}(\Omega)} := \max_{|\alpha|\le m} \|D^\alpha v\|_{L^\infty(\Omega)}. \tag{3.3}$$

We shall also write $\|\cdot\|_{m,p}$ in place of $\|\cdot\|_{W^{m,p}(\Omega)}$. Equipped with this norm or with an equivalent one (cf. Proposition 2.3), $W^{m,p}(\Omega)$ is called a **Sobolev space of order** m (and of integrability p).

• **Proposition 3.1** *For any $m \in \mathbf{N}$ and any $p \in [1,+\infty]$ the following occurs:*
(i) $W^{m,p}(\Omega)$ is a Banach space over $\mathbf{C}$.
(ii) If $1 \le p < +\infty$, $W^{m,p}(\Omega)$ is separable.
(iii) If $1 < p < +\infty$, $W^{m,p}(\Omega)$ is uniformly convex (hence reflexive).
(iv) $\|\cdot\|_{m,2}$ is a Hilbert norm. $W^{m,2}(\Omega)$ (which is usually denoted by $H^m(\Omega)$) is then a Hilbert space, equipped with the scalar product

$$(u,v) := \sum_{|\alpha|\le m} \int_\Omega D^\alpha u\, \overline{D^\alpha v}\, dx \qquad \forall u,v \in W^{m,2}(\Omega). \tag{3.4}$$

(v) If $p \ne \infty$, then for any $F \in W^{m,p}(\Omega)'$ there exists a family $\{f_\alpha\}_{|\alpha|\le m} \subset L^{p'}(\Omega)$ such that

$$\langle F, v\rangle = \sum_{|\alpha|\le m} \int_\Omega f_\alpha D^\alpha v\, dx \qquad \forall v \in W^{m,p}(\Omega). \tag{3.5}$$

This entails that

$$\|F\|_{W^{m,p}(\Omega)'} = \Big(\sum_{|\alpha|\le m} \|f_\alpha\|^{p'}_{L^{p'}(\Omega)} \Big)^{1/p'} \qquad \text{if } p \in]1,+\infty[, \tag{3.6}$$

$$\|F\|_{W^{m,1}(\Omega)'} = \max_{|\alpha|\le m} \|f_\alpha\|_{L^\infty(\Omega)}. \tag{3.7}$$

Conversely, for any family $\{f_\alpha\}_{|\alpha|\le m}$ as above, (3.5) defines a functional $F \in W^{m,p}(\Omega)'$. []

Proof. Parts (ii)—(v) can easily be proved by applying Proposition 2.3. [Ex] □

Extension Operators. We call a linear operator $E : L^1_{\rm loc}(\Omega) \to L^1_{\rm loc}(\mathbf{R}^N)$ a **regular extension operator** iff $Eu = u$ a.e. in Ω for any $u \in L^1_{\rm loc}(\mathbf{R}^N)$, and its restriction is continuous from $W^{m,p}(\Omega)$ to $W^{m,p}(\mathbf{R}^N)$ for any $m \in \mathbf{N}$ and any $p \in [1,+\infty]$. For instance the *trivial extension*

$$\tilde u := u \quad \text{in } \Omega, \qquad \tilde u := 0 \quad \text{in } \mathbf{R}^N \setminus \Omega, \tag{3.8}$$

is not a regular extension operator, whenever Ω is regular enough. For instance, if Ω is a ball then $u \equiv 1 \in W^{1,p}(\Omega)$, but obviously $\tilde u \notin W^{1,p}(\mathbf{R}^N)$.

• **Theorem 3.2** *(Calderón-Stein) For any uniformly-Lipschitz domain of $\mathbf{R}^N$, there exists a regular extension operator. []*

We illustrate the necessity of assuming some regularity for the domain Ω by means of two counterexamples.

Example 3.1. Let us set $Q :=]0,1[^2$, fix any $\lambda \in]0,1[$, and set

$$\Omega := \{(x,y) \in Q : y > x^\lambda\}, \qquad u_\gamma(x,y) := y^{-\gamma} \quad \forall (x,y) \in \Omega, \forall \gamma > 0. \tag{3.9}$$

For any $p \in [1,+\infty[$ a direct calculation shows that

$$u_\gamma \in W^{1,p}(\Omega) \quad \Leftrightarrow \quad p(\gamma+1) < 1 + \lambda^{-1}. \; [Ex] \tag{3.10}$$

Let us now assume that $(0 <)\gamma < (1+\lambda^{-1})/2-1$, namely $2(\gamma+1) < 1+\lambda^{-1}$; the inequality in (3.10) is then fulfilled by some $\tilde{p} > 2$. On the other hand $W^{1,\tilde{p}}(Q) \subset L^\infty(Q)$, by a result that we shall see in Sect. 3 (cf. Morrey's Theorem). Therefore the unbounded function u_γ cannot be extended to any element of $W^{1,\tilde{p}}(Q)$.

This example shows that, even for bounded domains, in Theorem 3.2 the hypothesis of Lipschitz regularity of Ω cannot be replaced by the uniform $C^{0,\lambda}$-regularity for any $\lambda \in]0,1[$. Note that for $\lambda = 1$ this construction fails, and actually in that case the Calderón-Stein Theorem 3.2 applies.

Example 3.2. Let us set

$$\Omega := \{(x,y) \in \mathbf{R}^2 : |x|, |y| < 1, x \neq 0\}, \qquad u : \Omega \to \mathbf{R} : (x,y) \mapsto \operatorname{sign}(x); \tag{3.11}$$

notice that $u \in W^{m,p}(\Omega)$ for any $m \in \mathbf{N}$ (actually, $u \in W^{m,p}(\Omega) \cap C^\infty(\Omega)$!), but it cannot be extended to any $w \in W^{m,p}(\mathbf{R}^2)$ for any $m \geq 1$. Actually Ω fulfills the cone property, but is not of class $C^{0,1}$.

Extension results are often applied to generalize to $W^{m,p}(\Omega)$ properties that are known to hold for $W^{m,p}(\mathbf{R}^N)$. As the restriction operator is obviously continuous from $W^{m,p}(\mathbf{R}^N)$ to $W^{m,p}(\Omega)$, under the hypotheses of Theorem 3.2, $W^{m,p}(\Omega)$ consists exactly of the restriction of the functions of $W^{m,p}(\mathbf{R}^N)$. The next statement then follows.

Corollary 3.3 *Let Ω be a uniformly-Lipschitz domain of $\mathbf{R}^N$. For any $m \in \mathbf{N}$ and any $p \in [1,+\infty]$, one can equip $W^{m,p}(\Omega)$ with the equivalent quotient norm*

$$\|v\| := \inf\{\|w\|_{W^{m,p}(\mathbf{R}^N)} : w \in W^{m,p}(\mathbf{R}^N), w|_\Omega = v\} \qquad \forall v \in W^{m,p}(\Omega). \quad [Ex] \tag{3.12}$$

- **Theorem 3.4** *(Density) Let $m \in \mathbf{N}$ and $p \in [1,+\infty[$.*
  (i) (Meyers and Serrin) For any domain Ω of $\mathbf{R}^N$, $C^\infty(\Omega) \cap W^{m,p}(\Omega)$ is dense in $W^{m,p}(\Omega)$.
  (ii) If Ω is uniformly-Lipschitz, then $\mathcal{D}(\bar{\Omega})$ is dense in $W^{m,p}(\Omega)$. [] [2]

It is easy to see that for $p = \infty$ both statements fail.

As for the regularity of Ω, it is easily seen that part (i) holds for the functions defined in Examples 3.1 and 3.2, whereas part (ii) fails.

Proposition 3.5 *(Calculus Rules) Let Ω be any domain of $\mathbf{R}^N$ and $p \in [1,+\infty]$.*
(i) For any $u,v \in W^{1,p}(\Omega) \cap L^{p'}(\Omega)$,

$$uv \in W^{1,1}(\Omega), \qquad \nabla(uv) = (\nabla u)v + u\nabla v \qquad \text{a.e. in } \Omega. \tag{3.13}$$

(ii) For any Lipschitz-continuous function $F : \mathbf{C} \to \mathbf{C}$ and any $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$, [2]

$$F(u) \in W^{1,p}_{\mathrm{loc}}(\Omega), \qquad \nabla F(u) = F'(u)\nabla u \qquad \text{a.e. in } \Omega. \tag{3.14}$$

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[2] By $\mathcal{D}(\bar{\Omega})$ we denote the space of restrictions to Ω of functions of $\mathcal{D}(\mathbf{R}^N)$. Equivalently, $\mathcal{D}(\bar{\Omega})$ is the space of functions $\Omega \to \mathbf{C}$ that can be extended to elements of $\mathcal{D}(\mathbf{R}^N)$.

[2] We set $W^{1,p}_{\mathrm{loc}}(\Omega) := \{v \in \mathcal{D}'(\Omega) : \varphi v \in W^{1,p}(\Omega), \forall \varphi \in \mathcal{D}(\Omega)\}$. Like $L^p_{\mathrm{loc}}(\Omega)$, this is not a normed space.

By Theorem 3.4(i) both statements can be proved via regularization. [Ex]

For any $h \in \mathbf{R}^N$ and any $\Omega \subset \mathbf{R}^N$, let us denote by τ_h the shift operator $v \mapsto v(\cdot + h)$.

Theorem 3.6 *For any* $p \in [1, +\infty]$,

$$v \in W^{1,p}(\mathbf{R}^N) \quad \Rightarrow \quad \|\tau_h v - v\|_{L^p(\mathbf{R}^N)} \le |h| \|\nabla v\|_{L^p(\mathbf{R}^N)^N} \quad \forall h \in \mathbf{R}^N. \tag{3.15}$$

The converse also holds if $p > 1$; that is, $v \in W^{1,p}(\mathbf{R}^N)$ whenever there exists a constant $C > 0$ such that for any $h \in \mathbf{R}^N$, $\|\tau_h v - v\|_{L^p(\mathbf{R}^N)} \le C|h|$. [] It is easily seen that this converse statement fails for $p = 1$ and $v = H$ (the Heaviside function).

* *Proof.* For $p = \infty$ the result is obvious; let us then assume that $p < +\infty$. By the Jensen inequality we have

$$|\tau_h v(x) - v(x)|^p = \Big| \int_0^1 h \cdot \nabla v(x + th)\, dt \Big|^p \le |h|^p \int_0^1 |\nabla v(x + th)|^p\, dt \qquad \text{for a.e. } x \in \mathbf{R}^N;$$

hence

$$\begin{aligned} &\|\tau_h v - v\|^p_{L^p(\mathbf{R}^N)} \le |h|^p \int_{\mathbf{R}^N} dx \int_0^1 |\nabla v(x + th)|^p\, dt \\ &= |h|^p \int_0^1 dt \int_{\mathbf{R}^N} |\nabla v(x + th)|^p\, dx = |h|^p \int_0^1 dt \int_{\mathbf{R}^N} |\nabla v(x)|^p\, dx = |h|^p \int_{\mathbf{R}^N} |\nabla v(x)|^p\, dx. \end{aligned} \qquad \square$$

4. Sobolev Spaces of Real Order and Traces

By part (ii) of Theorem 3.4, $\mathcal{D}(\mathbf{R}^N)$ is dense in $W^{m,p}(\mathbf{R}^N)$ for any $p \in [1, +\infty[$ and any $m \ge 1$. This holds for no other domain of class C^0; we just illustrate this issue via a simple example.

Let Ω be an open ball of $\mathbf{R}^N$, and set $u \equiv 1$ in Ω; obviously $u \in W^{m,p}(\Omega)$ for any $m \ge 1$ and any $p \in [1, +\infty[$. By contradiction, let us assume that it is possible to approximate u in the topology of $W^{m,p}(\Omega)$ by means of a sequence $\{u_n\} \subset \mathcal{D}(\Omega)$. The trivial extension operator $v \mapsto \tilde{v}$ (cf. (3.8)) is obviously continuous from $\mathcal{D}(\Omega)$ to $\mathcal{D}(\mathbf{R}^N)$ w.r.t. the $W^{m,p}$-topologies, for it obviously maps Cauchy sequences to Cauchy sequences; hence $\tilde{u}_n \to \tilde{u}$ in $W^{m,p}(\mathbf{R}^N)$. But we saw that $\tilde{u} \notin W^{m,p}(\mathbf{R}^N)$. Thus $\mathcal{D}(\Omega)$ is not dense in $W^{m,p}(\Omega)$.

On account of this negative result, we set

$$W_0^{m,p}(\Omega) := \text{ closure of } \mathcal{D}(\Omega) \text{ in } W^{m,p}(\Omega) \qquad \forall m \in \mathbf{N}, \forall p \in [1, +\infty[, \tag{4.1}$$

for any domain $\Omega \subset \mathbf{R}^N$, and equip this space with (the restriction of) the norm of $W^{m,p}(\Omega)$. The properties of Proposition 3.1 also hold for $W_0^{m,p}(\Omega)$, which indeed is a closed subspace of $W^{m,p}(\Omega)$. From this discussion we infer that $\Omega = \mathbf{R}^N$ is the only domain of class C^0 such that $W_0^{m,p}(\Omega) = W^{m,p}(\Omega)$ for any $m > 0$.

By the next statement, for any $m > 1$ the functions of $W_0^{m,p}(\Omega)$ may be regarded as vanishing on $\partial\Omega$ jointly with their derivatives up to order $m - 1$. (Under suitable regularity assumptions for Ω, this property might be restated in terms of *traces* — a notion that we introduce ahead.)

Proposition 4.1 *If m is a positive integer and the domain Ω is of class C^m, then*

$$(D^\alpha v)\big|_{\partial\Omega} = 0 \qquad \forall v \in W_0^{m,p}(\Omega) \cap C^{m-1}(\bar{\Omega}), \forall \alpha \in \mathbf{N}^N, |\alpha| \le m - 1.$$

Sobolev Spaces of Negative Order. Next we set

$$W^{-m,p'}(\Omega) := W_0^{m,p}(\Omega)' \ \ (\subset \mathcal{D}'(\Omega)) \qquad \forall m \in \mathbf{N}, \forall p \in [1,+\infty[, \tag{4.2}$$

and equip it with the dual norm

$$\|u\|_{W^{-m,p'}(\Omega)} := \sup\big\{\langle u,v\rangle : v \in W_0^{m,p}(\Omega), \|v\|_{W^{m,p}(\Omega)} = 1\big\}$$

(here by $\langle \cdot,\cdot \rangle$ we denote the pairing between $W^{-m,p'}(\Omega)$ and $W_0^{m,p}(\Omega)$). [2]

The Sobolev spaces of negative order inherit several properties from their preduals.

Proposition 4.2 *For any $m \in \mathbf{N}$ and any $p \in [1,+\infty[$, $W^{-m,p'}(\Omega)$ is a Banach space.*

(i) If $1 < p < +\infty$, $W^{-m,p'}(\Omega)$ is separable and reflexive.

(ii) $\|\cdot\|_{-m,2}$ is a Hilbert norm, and $W^{-m,2}(\Omega)$ is a Hilbert space (that is usually denoted by $H^{-m}(\Omega)$).

Proposition 4.3 *(Characterization of Sobolev Spaces of Negative Integer Order) For any $m \in \mathbf{N}$ and any $p \in [1,+\infty[$,*

$$F \in W^{-m,p'}(\Omega) \quad \Leftrightarrow \quad \exists \{f_\alpha\}_{|\alpha|\le m} \subset L^{p'}(\Omega):\ F = \textstyle\sum_{|\alpha|\le m} D^\alpha f_\alpha \quad \text{in } \mathcal{D}'(\Omega). \tag{4.3}$$

[This representation of F need not be unique.]

Proof. By the Hahn-Banach theorem any $F \in W^{-m,p'}(\Omega)$ can be extended to a functional $\tilde{F} \in W^{m,p}(\Omega)'$. By part (v) of Proposition 3.1 then there exists a family $\{f_\alpha\}_{|\alpha|\le m}$ in $L^{p'}(\Omega)$ such that

$$\langle \tilde{F}, v\rangle = \sum_{|\alpha|\le m} (-1)^{|\alpha|} \int_\Omega f_\alpha D^\alpha v \, dx \qquad \forall v \in W^{m,p}(\Omega),$$

Restricting this equality to $v \in \mathcal{D}(\Omega)$, we then get $F = \sum_{|\alpha|\le m} D^\alpha f_\alpha$ in $\mathcal{D}'(\Omega)$.

Conversely, any distribution of this form is obviously a functional of $W^{-m,p'}(\Omega)$. □

Sobolev Spaces of Positive Noninteger Order. Let us fix any $p \in [1,+\infty[$, any $\lambda \in]0,1[$, set

$$[a_{\lambda,p}(v)](x,y) := \frac{v(x)-v(y)}{|x-y|^{\frac{N}{p}+\lambda}} \qquad \forall x,y \in \Omega\,(x \ne y), \forall v \in L^1_{\rm loc}(\Omega), \tag{4.4}$$

$$W^{\lambda,p}(\Omega) := \big\{v \in L^p(\Omega) : a_{\lambda,p}(v) \in L^p(\Omega^2)\big\}, \tag{4.5}$$

and equip this space with the p-norm of the graph

$$\|v\|_{\lambda,p} := \big(\|v\|^p_{L^p(\Omega)} + \|a_{\lambda,p}(v)\|^p_{L^p(\Omega^2)}\big)^{1/p}. \tag{4.6}$$

In order to complete this picture we also set

$$W^{\lambda,\infty}(\Omega) := C^{0,\lambda}(\bar{\Omega}) \qquad \forall \lambda \in]0,1[. \tag{4.7}$$

Let us next fix any positive $m \in \mathbf{N}$, and set

$$W^{m+\lambda,p}(\Omega) := \big\{v \in W^{m,p}(\Omega) : D^\alpha v \in W^{\lambda,p}(\Omega),\ \forall \alpha \in \mathbf{N}^N, |\alpha| = m\big\}; \tag{4.8}$$

---

[2] Notice that we have thus defined $W^{-m,q}(\Omega)$ only for $1 < q \le +\infty$, and that for $m = 0$ we retrieve $W^{0,p'}(\Omega) = L^{p'}(\Omega)$.

this is a normed space over $\mathbf{C}$ equipped with the p-norm of the graph

$$\begin{aligned}\|v\|_{m+\lambda,p} &:= \Big(\|v\|^p_{m,p} + \sum_{|\alpha|=m} \|D^\alpha v\|^p_{\lambda,p}\Big)^{1/p} \\ &= \Big(\sum_{|\alpha|\le m} \int_\Omega |D^\alpha v|^p\,dx + \sum_{|\alpha|=m} \iint_{\Omega^2} |[a_{\lambda,p}(D^\alpha v)](x,y)|^p\,dxdy\Big)^{1/p}.\end{aligned} \tag{4.9}$$

Let us also set

$$W^{m+\lambda,\infty}(\Omega) := C^{m,\lambda}(\bar\Omega) \qquad \forall m\in\mathbf{N}, \forall\lambda\in\,]0,1[. \tag{4.10}$$

The spaces $W^{m+\lambda,p}(\Omega)$ are called *Sobolev spaces of fractional order* (sometimes just *fractional Sobolev spaces),* or also *Slobodeckiĭ spaces.*

Proposition 4.4 *For any $m\in\mathbf{N}$, any $0<\lambda<1$ and any $p\in[1,+\infty[$, setting $s:=m+\lambda$ the following occurs:*

(i) $W^{s,p}(\Omega)$ is a Banach space over $\mathbf{C}$. For any $q\in[1,+\infty]$, $W^{s,p}(\Omega)$ may also be equipped with the q-norm of the graph, *and these norms are equivalent.*

(ii) If $p<+\infty$, $W^{s,p}(\Omega)$ is separable.

(iii) If $1<p<+\infty$, $W^{s,p}(\Omega)$ is uniformly convex (hence reflexive).

(iv) $\|\cdot\|_{s,2}$ is a Hilbert norm. $W^{s,2}(\Omega)$ (that will be denoted by $H^s(\Omega)$) is a Hilbert space, equipped with the scalar product

$$(u,v) := \sum_{|\alpha|\le m} \int_\Omega D^\alpha u(x)\,\overline{D^\alpha v}(x)\,dx + \sum_{|\alpha|=m} \iint_{\Omega^2} \frac{D^\alpha u(x)\,\overline{D^\alpha v(y)}}{|x-y|^{N+2\lambda}}\,dxdy \qquad \forall u,v\in W^{s,2}(\Omega). \tag{4.11}$$

Outline of the Proof. If $p=+\infty$ we already know that $W^{m+\lambda,\infty}(\Omega) := C^{m,\lambda}(\bar\Omega)$ is a Banach space. If $p<+\infty$, we set

$$L_1(v) := \{D^\alpha v : |\alpha|\le m\}, \quad L_2(v) := \{a_{\lambda,p}(D^\alpha v) : |\alpha| = m\} \qquad \forall v\in L^p(\Omega);$$

the thesis then follows by applying Proposition 2.2. □

Proposition 4.5 *Let Ω be any nonempty domain of $\mathbf{R}^N$, and set $\Omega_n := \{x\in\Omega : d(x,\mathbf{R}^N\setminus\Omega) > 1/n\}$ for any $n\in\mathbf{N}$. Then*

$$\|u\|_{W^{s,p}(\Omega_n)} \to \|u\|_{W^{s,p}(\Omega)} \qquad \forall u\in W^{s,p}(\Omega), \forall s\ge 0, \forall p\in[1,+\infty]. \tag{4.12}$$

Outline of the Proof. With no loss of generality one may assume that Ω is bounded. For $p\neq\infty$, the statement then follows from the absolute continuity of the integral. For $p=\infty$ the proof is even simpler. [Ex] □

Hilbert-Type Sobolev Spaces of Fractional Order. For any $u\in L^1$, [3] we set

$$\tilde H^s := \{v\in\mathcal{S}' : (1+|\xi|^2)^{s/2}\hat v\in L^2\} \qquad \forall s\in\mathbf{R}, \tag{4.13}$$

which is a Hilbert space equipped with the norm

$$\|v\|_{\tilde H^s} := \|(1+|\xi|^2)^{s/2}\hat v\|_{L^2} \qquad \forall v\in\tilde H^s. \tag{4.14}$$

(3) We still write L^1 instead of $L^1(\mathbf{R}^N)$ and similarly, and denote the *Fourier transform* of any $u\in\mathcal{S}'$ by $\hat u$.

By the Plancherel theorem

$$\int_{\mathbf{R}^N} uv\,dx = \int_{\mathbf{R}^N} \hat{u}\hat{v}\,dx = \int_{\mathbf{R}^N} [(1+|\xi|^2)^{s/2}\hat{u}]\,[(1+|\xi|^2)^{-s/2}\hat{v}]\,dx \\ \le \|u\|_{\tilde{H}^s}\,\|v\|_{\tilde{H}^{-s}} \qquad \forall u,v\in\mathcal{S}, \forall s\in\mathbf{R}. \tag{4.15}$$

One can then show that $(\tilde{H}^s)'$ can be identified with $\tilde{H}^{-s}$ for any real s.

Proposition 4.6 *For any $s\in\mathbf{R}$, $\tilde{H}^s = H^s$.*

For any $m\in\mathbf{Z}$, the equivalence between the norms of $\tilde{H}^m$ and H^m is easily checked, since for any $\alpha\in\mathbf{N}^N$ $\mathcal{F}(D^\alpha u) = (2\pi i\xi)^\alpha\hat{u}$, whence by the Plancherel theorem

$$\|D^\alpha u(x)\|_{L^2} = \|\mathcal{F}[D^\alpha u]\|_{L^2} = (2\pi)^{|\alpha|}\|\xi^\alpha\hat{u}\|_{L^2}.$$

Moreover

$$\exists C>0 : \forall\xi\in\mathbf{R}^N \qquad C(1+|\xi|^2)^{|\alpha|/2} \le 1+|\xi|^{|\alpha|} \le (1+|\xi|^2)^{|\alpha|/2}. \qquad [Ex]$$

The Hilbert spaces $\tilde{H}^s$ (with $s\in\mathbf{R}$) may thus be regarded as Sobolev spaces (of real order).

The definition of the spaces $\tilde{H}^s$ $(=\tilde{H}^s(\mathbf{R}^N))$ is then extended to $\tilde{H}^s(\Omega)$ for any uniformly Lipschitz domain $\Omega\subset\mathbf{R}^N$ as follows, in analogy with (3.12):

$$\tilde{H}^s(\Omega) = \{w|_\Omega : w\in\tilde{H}^s(\mathbf{R}^N)\}, \\ \|v\|_{\tilde{H}^s(\Omega)} = \inf\{\|w\|_{\tilde{H}^s(\mathbf{R}^N)} : w|_\Omega = v\} \quad \forall s>0. \tag{4.16}$$

Therefore $\tilde{H}^s(\Omega) = H^s(\Omega)$ $(=W^{s,2}(\Omega))$ for any $s\in\mathbf{R}$.

Sobolev Spaces of Negative Noninteger Order. This construction mimics that of Sobolev spaces of negative integer order. First we set

$$W_0^{s,p}(\Omega) := \text{ closure of } \mathcal{D}(\Omega) \text{ in } W^{s,p}(\Omega) \qquad \forall s\ge 0, \forall p\in[1,+\infty[, \tag{4.17}$$

and equip it with the topology induced by $W^{s,p}(\Omega)$. The properties stated in Proposition 3.1 hold also for $W_0^{s,p}(\Omega)$. (4) This is a normal space of distributions, hence its dual is also a space of distributions. We then set

$$W^{-s,p'}(\Omega) := W_0^{s,p}(\Omega)' \;\; (\subset\mathcal{D}'(\Omega)) \qquad \forall s\ge 0, \forall p\in[1,+\infty[, \tag{4.18}$$

and equip it with the dual norm

$$\|u\|_{-s,p'} := \sup\{\langle u,v\rangle : v\in W_0^{s,p}(\Omega), \|v\|_{s,p} = 1\}.$$

A result analogous to Proposition 4.2 holds for $W^{-s,p'}(\Omega)$.

We have thus completed the definition of the *scale* of Sobolev spaces. In the next statement we gather their main properties.

Proposition 4.7 *Let $s\in\mathbf{R}$ and $p\in]1,+\infty]$ (with $p=1$ included if $s\ge 0$). Then:*
(i) $W^{s,p}(\Omega)$ is a Banach space over $\mathbf{C}$.
(ii) If $p<+\infty$, $W^{s,p}(\Omega)$ is separable.
(iii) If $1<p<+\infty$, $W^{s,p}(\Omega)$ is reflexive.

(4) Theorems 3.2—3.4 also hold for fractional indices, too. []

(iv) $\|\cdot\|_{s,2}$ *is a Hilbert norm, and* $W^{s,2}(\Omega)$ *(=:* $H^s(\Omega)$*) is a Hilbert space.*
(v) If $s \geq 0$*, the same properties hold for* $W_0^{s,p}(\Omega)$*, the closure of* $\mathcal{D}(\Omega)$ *in* $W^{s,p}(\Omega)$*.*

Let us set

$$W^{s,p}_{\rm loc}(\Omega) := \big\{v \in \mathcal{D}'(\Omega) : \varphi v \in W^{s,p}(\Omega), \forall \varphi \in \mathcal{D}(\Omega)\big\} \qquad \forall s \in \mathbf{R}, \forall p \in [1,+\infty]. \tag{4.19}$$

This is a Fréchet space, equipped with the family of seminorms $\{v \mapsto \|\varphi v\|_{s,p} : \varphi \in \mathcal{D}(\Omega)\}$; indeed this topology can be generated by a countable family of these seminorms.

Traces. Dealing with PDEs it is of paramount importance to prescribe boundary- (and/or initial-) values. However, for functions of Sobolev spaces the restriction to a lower dimensional manifold $\mathcal{M} \subset \bar\Omega$ is meaningless, for $\mathcal{M}$ has vanishing Lebesgue measure and these functions are only defined a.e. in Ω. Nevertheless by means of functional methods one can generalize the concept of *restriction* by introducing the notion of *trace.*

For instance, let $\Omega :=]0,1[$ and $\mathcal{M} := \{x_0\}$. For any $v \in C^1([0,1])$ and any $x, x_0 \in [0,1]$, we have $v(x_0) = v(x) + \int_x^{x_0} v'(\xi)\,d\xi$; hence

$$|v(x_0)| = \int_0^1 |v(x_0)|\,dx \leq \int_0^1 \Big(|v(x)| + \int_x^{x_0} |v'(\xi)|\,d\xi\Big)\,dx \leq \|v\|_{W^{1,1}(0,1)}.$$

The restriction $v \mapsto v(x_0)$ may thus be extended to a uniquely defined continuous operator $W^{1,1}(0,1) \to \mathbf{R}$. Let us now set $\Omega :=]0,1[^2$. By a similar argument, one can easily check that $v(0,\cdot) \in L^p(0,1)$ whenever $v, D_{x_1}v \in L^p(\Omega)$, and moreover, for a suitable constant $C > 0$,

$$\|v(0,\cdot)\|_{L^p(0,1)} \leq C\big(\|v\|_{L^p(\Omega)} + \|D_{x_1}v\|_{L^p(\Omega)}\big) \qquad \text{if } v, D_{x_1}v \in L^p(\Omega).$$

Next we state two basic trace results. First notice that Γ may be equipped with the $(N-1)$-dimensional Hausdorff measure whenever Ω is (e.g.) of class $C^{0,1}$. One can then define the Banach space $L^p(\Gamma)$ for any $p \in [1,+\infty]$.

• **Theorem 4.8** *(Traces) Let* $1 \leq p \leq +\infty$*,* $s > 1/p$*, and* Ω *be a bounded domain of* $\mathbf{R}^N$ *of class* $C^{0,1}$*. There exists a (unique) linear and continuous* trace operator $\gamma_0 : W^{s,p}(\Omega) \to L^p(\Gamma)$ *such that* $\gamma_0 v = v$ *on* Γ *for any* $v \in \mathcal{D}(\bar\Omega)$*. []*

• **Theorem 4.9** *(Normal Traces) Let* $1 \leq p \leq +\infty$*,* $s > 1 + 1/p$*,* Ω *be a bounded domain of* $\mathbf{R}^N$ *of class* $C^{1,1}$*, and* $\vec\nu$ *be the outward-oriented unit normal vector field on* Γ*. There exists a (unique) linear and continuous* normal trace operator $\gamma_1 : W^{s,p}(\Omega) \to L^p(\Gamma)$ *such that* $\gamma_1 v = \partial v/\partial\vec\nu$ *on* Γ *for any* $v \in \mathcal{D}(\bar\Omega)$*. []*

Notice that the normal trace is the trace of the derivative in the normal direction: $\gamma_1 v = \gamma_0(\vec\nu\cdot\nabla v)$, after that the normal field has been smoothly extended to a neighbourhood of $\partial\Omega$.

We can also characterize the spaces $W_0^{1,p}$ and $W_0^{2,p}$ in terms of traces (cf. Proposition 4.1):

• **Proposition 4.9** *Let* Ω *be a bounded domain of* $\mathbf{R}^N$ *of class* $C^{1,1}$*. For any* $p \in [1,+\infty]$*,*

$$W_0^{1,p}(\Omega) = \{v \in W^{1,p}(\Omega) : \gamma_0 v = 0 \ \ \text{a.e. on } \Gamma\}, \tag{4.20}$$
$$W_0^{2,p}(\Omega) = \{v \in W^{2,p}(\Omega) : \gamma_1 v = \gamma_0 v = 0 \ \ \text{a.e. on } \Gamma\}. \tag{4.21}$$

More generally, for $k = 1,2$, $W_0^{k,p}(\Omega)$ is the space of all functions of $W^{k,p}(\Omega)$ such that all the traces that make sense in $W^{k,p}(\Omega)$ vanish. [] Thus for instance

$$W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) = \{v \in W^{2,p}(\Omega) : \gamma_0 v = 0 \ \text{ a.e. on } \Gamma\} \neq W_0^{2,p}(\Omega). \tag{4.22}$$

The next result is often applied in the study of PDEs with Dirichlet boundary conditions.

Theorem 4.10 *(Friedrichs Inequality) Assume that Ω is a bounded Lipschitz domain of $\mathbf{R}^N$, let $\Gamma_1 \subset \Gamma$ have positive $(N-1)$-dimensional measure, and $p \in [1, +\infty]$. Then*

$$v \mapsto \|v\| := \big(\|\nabla v\|^p_{L^p(\Omega)^N} + \|\gamma_0 v\|^p_{L^p(\Gamma_1)}\big)^{1/p} \tag{4.23}$$

is an equivalent norm in $W^{1,p}(\Omega)$.

* *Proof.* By the continuity of the trace operator $W^{1,p}(\Omega) \to L^p(\Gamma_1)$, there exists $C > 0$ such that $\|v\| \le C\|v\|_{1,p}$ for any $v \in W^{1,p}(\Omega)$. The converse inequality holds if we show that there exists $\hat{C} > 0$ such that

$$\|v\|_{L^p(\Omega)} \le \hat{C}\big(\|\nabla v\|^p_{L^p(\Omega)^N} + \|\gamma_0 v\|^p_{L^p(\Gamma_1)}\big)^{1/p} \qquad \forall v \in W^{1,p}(\Omega).$$

By contradiction, let us assume that for any $n \in \mathbf{N}$ there exists $v_n \in W^{1,p}(\Omega)$ such that

$$\|v_n\|_{L^p(\Omega)} > n\big(\|\nabla v_n\|^p_{L^p(\Omega)^N} + \|\gamma_0 v_n\|^p_{L^p(\Gamma_1)}\big)^{1/p}. \tag{4.24}$$

Possibly dividing this inequality by $\|v_n\|_{L^p(\Omega)}$, we can assume that $\|v_n\|_{L^p(\Omega)} = 1$ for any n. We infer that there exists $v \in W^{1,p}(\Omega)$ such that, possibly extracting a subsequence, $v_n \to v$ weakly in $W^{1,p}(\Omega)$. By (4.24), $\nabla v_n \to 0$ strongly in $L^p(\Omega)^N$ and $\gamma_0 v_n \to 0$ strongly in $L^p(\Gamma_1)$. Hence $\nabla v = 0$ a.e. in Ω and $\gamma_0 v = 0$ a.e. on Γ_1. This entails that $v = 0$ a.e. in Ω, as Ω is connected. On the other hand, as the injection $W^{1,p}(\Omega) \to L^p(\Omega)$ is compact, [5] $\|v\|_{L^p(\Omega)} = \lim_{n\to+\infty} \|v_n\|_{L^p(\Omega)} = 1$, and this is a contradiction. □

Exercises.

— Let Ω be a uniformly-Lipschitz domain of $\mathbf{R}^N$ and $1 \le p \le +\infty$. For any $s \in \mathbf{R}$, let us denote by $W^{s,p}_c(\Omega)$ the subspace of compactly supported distributions of $W^{s,p}(\Omega)$. Show that

$$\bigcap_{s\in\mathbf{R}} W^{s,p}_c(\Omega) = \mathcal{D}(\Omega), \quad \bigcup_{s\in\mathbf{R}} W^{s,p}_c(\Omega) = \mathcal{E}'(\Omega), \quad \bigcap_{s\in\mathbf{R}} W^{s,p}_{loc}(\Omega) = \mathcal{E}(\Omega), \quad \bigcup_{s\in\mathbf{R}} W^{s,p}_{loc}(\Omega) = \mathcal{D}'_F(\Omega)$$

(the latter is the space of distributions of finite order).

5. Sobolev and Morrey Imbeddings

Basic Imbeddings. Obviously

$$|\Omega| < +\infty \quad \Rightarrow \quad C^m(\bar{\Omega}) \subset W^{m,p}(\Omega) \qquad \forall m \in \mathbf{N}, \forall p \in [1, +\infty], \tag{5.1}$$

whereas the opposite inclusion fails, and $C^{m,1}(\bar{\Omega}) \subset W^{m+1,\infty}(\Omega)$ for any domain Ω. Moreover

$$\Omega \in C^0 \quad \Rightarrow \quad C^{m,1}(\bar{\Omega}) = W^{m+1,\infty}(\Omega) \qquad \forall m \in \mathbf{N}. \quad [] \tag{5.2}$$

The following simple counterexample shows that this equality fails if Ω is not regular enough. Let Ω_1 be as in (2.4XXX), and set $u(\rho, \theta) = \theta$ for any $(\rho, \theta) \in \Omega_1$. Then $u \in W^{m,p}(\Omega_1)$ for any $m \in \mathbf{N}$ and any $p \in [1, +\infty]$, but $u \notin C^0(\bar{\Omega}_1)$. Actually the domain Ω_1 fulfills the cone property but is not of class C^0.

[5] This property will be seen ahead...

For reasons that will appear ahead, we shall set

$$W^{m+\lambda,\infty}(\Omega) := C^{m,\lambda}(\bar{\Omega}) \qquad \forall m \in \mathbf{N}, \forall \lambda \in]0,1[, \forall \text{ domain } \Omega \subset \mathbf{R}^N. \tag{5.3}$$

Next we compare Sobolev spaces having either different differentiability indices, m, or different integrability indices, p. We shall see that the larger is either index, the smaller is the space (under appropriate hypotheses).

Proposition 5.1 *For any domain $\Omega \subset \mathbf{R}^N$, any $m \in \mathbf{N}$ and any $p_1, p_2 \in [1, +\infty]$,*

$$|\Omega| < +\infty,\ p_1 < p_2 \quad \Rightarrow \quad \begin{array}{l} W^{m,p_2}(\Omega) \subset W^{m,p_1}(\Omega) \\ W_0^{m,p_2}(\Omega) \subset W_0^{m,p_1}(\Omega) \end{array} \quad \textit{(with density)}. \tag{5.4}$$

Proof. (5.4) directly follow from the analogous inclusion between L^p-spaces. □

Proposition 5.2 *Let Ω be a uniformly-Lipschitz domain of $\mathbf{R}^N$. For any $m_1, m_2 \in \mathbf{N}$ and for any $p \in [1, +\infty]$,*

$$m_1 \leq m_2 \quad \Rightarrow \quad \begin{array}{l} W^{m_2,p}(\Omega) \subset W^{m_1,p}(\Omega) \\ W_0^{m_2,p}(\Omega) \subset W_0^{m_1,p}(\Omega) \end{array} \quad \textit{(with density)}. \tag{5.5}$$

Proof. The inclusions are obvious. As by Theorem 3.4 $\mathcal{D}(\bar{\Omega})$ is dense in both spaces, the density follows. □

The Sobolev Theorem. Two further classes of imbedding results are of paramount importance in the theory of Sobolev spaces, and respectively concern imbeddings between Sobolev spaces and from Sobolev to Hölder spaces:

$$W^{r,p}(\Omega) \subset W^{s,q}(\Omega) \quad \text{and} \quad W^{r,p}(\Omega) \subset C^{\ell,\lambda}(\bar{\Omega}) \quad \text{(for suitable indices)}.$$

These results are first proved for $\Omega = \mathbf{R}^N$ and then generalized to any uniformly-Lipschitz domain via Calderón-Stein's Theorem 3.2.

After Propositions 5.1 and 5.2 we already know that the larger are either the differentiability index m or the integrability index p or both, the smaller is $W^{m,p}(\Omega)$. What happens when these two indices vary in opposite directions? We shall see that, under appropriate restrictions on the integrability indices, the larger is m the smaller is the space. (Loosely speaking, the differentiability prevails over the integrability.) The converse always fails:

$$\begin{array}{c} m_1 < m_2 \quad \Rightarrow \quad W^{m_1,p}(\Omega) \not\subset W^{m_2,q}(\Omega) \\ \forall m_1, m_2 \in \mathbf{N}, \forall p, q \in [1, +\infty], \forall \Omega. \end{array} \qquad [Ex] \tag{5.7}$$

Nontrivial imbeddings between Sobolev spaces rest on the following fundamental inequality due to Sobolev.

• **Theorem 5.3** *(Sobolev Inequality) For any $N > 1$ and any $p \in [1, N[$, there exists a constant $C = C_{N,p} > 0$ such that, setting $p^* := Np/(N-p)$,*

$$\|u\|_{L^{p^*}(\mathbf{R}^N)} \leq C \|\nabla u\|_{L^p(\mathbf{R}^N)^N} \qquad \forall u \in \mathcal{D}(\mathbf{R}^N). \quad [] \tag{5.8}$$

Although this inequality only applies to functions with bounded support ($u \equiv 1$ is an obvious counterexample), the constant C does not depend on the support.

Proof for $p = 1$. In this case the argument is much simpler than in the general setting. We just illustrate the procedure for $N = 2$, however the extension to any N is trivial. For any $u \in \mathcal{D}(\mathbf{R}^2)$,

$$|u(x,y)| = \Big| \int_{-\infty}^{x} \frac{\partial u}{\partial \tilde{x}}(\tilde{x}, y)\, d\tilde{x} \Big| \leq \int_{\mathbf{R}} |\nabla u(\tilde{x}, y)|\, d\tilde{x} \qquad \forall (x,y) \in \mathbf{R}^2,$$

and similarly $|u(x,y)| \leq \int_{\mathbf{R}} |\nabla u(x, \tilde{y})|\, d\tilde{y}$. Therefore

$$\begin{aligned}\iint_{\mathbf{R}^2} |u(x,y)|^2\, dxdy &\leq \iint_{\mathbf{R}^2} \Big(\int_{\mathbf{R}} |\nabla u(\tilde{x}, y)|\, d\tilde{x} \Big) \Big(\int_{\mathbf{R}} |\nabla u(x, \tilde{y})|\, d\tilde{y} \Big)\, dxdy \\ &= \iint_{\mathbf{R}^2} |\nabla u(\tilde{x}, y)|\, d\tilde{x}dy \iint_{\mathbf{R}^2} |\nabla u(x, \tilde{y})|\, dxd\tilde{y} \\ &= \Big(\iint_{\mathbf{R}^2} |\nabla u(x,y)|\, dxdy \Big)^2,\end{aligned}$$

that is, $\|u\|_{L^2(\mathbf{R}^2)} \leq \|\nabla u\|_{L^1(\mathbf{R}^2)^2}$. Of course here $2^* = 1$ (for $N = 2$). □

Remark. If we assume that an inequality of the form (5.8) is fulfilled, we can establish the relation between p^* and p via the following simple *scaling argument.* Let us fix any $u \in \mathcal{D}(\mathbf{R}^N)$ and set $v_t(x) := u(tx)$ for any $x \in \mathbf{R}^N$ and any $t > 0$. Writing (5.8) for v_t we then get

$$\|u\|_{L^{p^*}(\mathbf{R}^N)} \leq C t^{1+N/p^* - N/p} \|\nabla u\|_{L^p(\mathbf{R}^N)^N} \qquad \forall u \in \mathcal{D}(\mathbf{R}^N), \forall t > 0.$$

This inequality may hold only if $1 + N/p^* - N/p = 0$, that is, $p^* := Np/(N-p)$.

Sobolev's Imbeddings. As obviously $\|\nabla u\|_{L^p(\mathbf{R}^N)^N} \leq \|u\|_{1,p}$, by the density of $\mathcal{D}(\mathbf{R}^N)$ in $W^{1,p}(\mathbf{R}^N)$ the Sobolev inequality (5.8) entails

$$W^{1,p}(\mathbf{R}^N) \subset L^{p^*}(\mathbf{R}^N) \ \big(=: W^{0,p^*}(\mathbf{R}^N) \big). \tag{5.9}$$

On this basis one can prove the following more general result.

• **Theorem 5.4** *(Sobolev Imbeddings) Let Ω be a uniformly-Lipschitz domain of $\mathbf{R}^N$. For any $\ell, m \in \mathbf{N}$ and any $p, q \in [1, +\infty]$,*

$$p \leq q, \ \ \ell - \frac{N}{q} \leq m - \frac{N}{p} \quad \Rightarrow \quad W^{m,p}(\Omega) \subset W^{\ell,q}(\Omega) \tag{5.10}$$

(with density if $q \neq +\infty$). This statement also holds if both W-spaces are replaced either by the corresponding W_0-spaces, or by the corresponding $W_{\rm loc}$-spaces. In the two latter cases Ω may be any domain of $\mathbf{R}^N$.

* *Proof.* On account of the regularity of Ω, by the Calderón-Stein's Theorem 3.2 it suffices to prove the inclusion for $\Omega = \mathbf{R}^N$. It also suffices to deal with $m = 1$ and $\ell = 0$, since by applying this result iteratively one may then get it in general.

The Sobolev inequality (3.7) yields $W^{1,p}(\mathbf{R}^N) \subset L^{p^*}(\mathbf{R}^N)$. Let $q \in [p, p^*]$; as $W^{1,p}(\mathbf{R}^N) \subset L^p(\mathbf{R}^N)$ and $L^p(\mathbf{R}^N) \cap L^{p^*}(\mathbf{R}^N) \subset L^q(\mathbf{R}^N)$, [Ex] we conclude that $W^{1,p}(\mathbf{R}^N) \subset L^q(\mathbf{R}^N)$.

We claim that the injection operator $j : W^{m,p}(\Omega) \to W^{\ell,q}(\Omega)$ is continuous. By the Closed Graph Theorem, it suffices to show that the set $G := \{(v, jv) : v \in W^{m,p}(\Omega)\}$ is closed in $W^{m,p}(\Omega) \times W^{\ell,q}(\Omega)$. Now, if $(v_n, jv_n) \to (v, w)$ in the latter space, then there exists a subsequence $\{v_{n'}\}$ such that $v_{n'} \to v$ a.e. in Ω. Hence $w = jv$ a.e. in Ω. □

Remarks. (i) $p \leq q$ and $\ell - N/q \leq m - N/p$ entail $\ell \leq m$, consistently with (5.7).

(ii) If $|\Omega| < +\infty$, then in (5.10) the hypothesis $p \le q$ may be replaced by $\ell \le m$. [Ex]

Morrey Imbeddings. Next we come to another important class of imbeddings, that read $W^{m,p}(\Omega) \subset C^{\ell,\lambda}(\bar\Omega)$ under suitable hypotheses on m, p, ℓ, λ. By an inclusion like this we mean that for any $v \in W^{m,p}(\Omega)$ there exists a (necessarily unique) $\hat v \in C^{\ell,\lambda}(\bar\Omega)$ such that $\hat v = v$ a.e. in Ω. That is, the equivalence class $W^{m,p}(\Omega)$ contains one (and only one) function of $C^{\ell,\lambda}(\bar\Omega)$. Henceforth we shall systematically assume this convention.

The next result applies to the case of $p > N$, which is not covered by Sobolev's Theorem 5.4.

• **Theorem 5.5** *(Morrey Imbeddings) Let Ω be a uniformly-Lipschitz domain of $\mathbf{R}^N$, $\ell, m \in \mathbf{N}$, $1 \le p < +\infty$ and $0 < \lambda < 1$. Then*

$$\ell + \lambda \le m - \frac{N}{p} \quad \Rightarrow \quad W^{m,p}(\Omega) \subset C^{\ell,\lambda}(\bar\Omega). \tag{5.11}$$

Moreover, [6]

$$W^{m+N,1}(\Omega) \subset C^m_b(\Omega).\ [] \tag{5.12}$$

Proof of (5.12). It suffices to show this statement for $\Omega = \mathbf{R}^N$ and for $m = 0$. We have

$$\begin{aligned} |u(x_1, \dots, x_N)| &= \left| \int_{-\infty}^{x_1} dy_1 \cdots \int_{-\infty}^{x_N} dy_N \, \frac{\partial^N u}{\partial y_1 \cdots \partial y_N}(y_1, \dots, y_N) \right| \\ &\le \left\| \frac{\partial^N u}{\partial y_1 \cdots \partial y_N} \right\|_{L^1(\mathbf{R}^N)} \le \|u\|_{N,1} \qquad \forall u \in \mathcal{D}(\mathbf{R}^N). \end{aligned}$$

By density we then get $\|u\|_{C^0_b(\mathbf{R}^N)} \le \|u\|_{N,1}$ for any $u \in W^{N,1}(\mathbf{R}^N)$. [Ex] □

Remarks. (i) In particular (5.12) yields $W^{1,1}(\Omega) \subset L^\infty(\Omega)$ if $N = 1$. However

$$W^{1,N}(\Omega) \not\subset L^\infty(\Omega) \qquad \forall N > 1. \tag{5.13}$$

For instance, setting $\Omega := B(0, 1/2)$ (the ball of center the origin and radius 2) and

$$v_\alpha(x) := (-\log|x|)^\alpha \qquad \forall x \in \Omega, \forall \alpha \in \,]0, 1 - 1/N[, \tag{5.14}$$

it is easy to check that $v_\alpha \in W^{1,N}(\Omega))$, although of course $v_\alpha \notin L^\infty(\Omega)$.

(ii) After (5.3), setting $N/\infty := 0$ the Morrey imbedding (5.11) might be regarded as a limit case of the Sobolev imbedding (5.9) for $q = \infty$.

Regularity Indices. Defining

the **Sobolev index** $\quad I_S(m,p) := m - N/p \quad$ for the Sobolev space $W^{m,p}(\Omega)$, (5.15)

the **Hölder index** $\quad I_H(m,\lambda) := m + \lambda \quad$ for the Hölder space $C^{m,\lambda}(\bar\Omega)$, (5.16)

the implications (5.10) and (5.11) of the Sobolev and Morrey imbeddings respectively also read

$$p \le q,\ \ell \le m,\ I_S(\ell, q) \le I_S(m,p) \quad \Rightarrow \quad W^{m,p}(\Omega) \subset W^{\ell,q}(\Omega), \tag{5.17}$$

$$\ell \le m,\ I_H(\ell, \lambda) \le I_S(m,p) \quad \Rightarrow \quad W^{m,p}(\Omega) \subset C^{\ell,\lambda}(\bar\Omega). \tag{5.18}$$

• **Theorem 5.6** *(Compactness) Let Ω be a bounded Lipschitz domain of $\mathbf{R}^N$, $\ell, m \in \mathbf{N}_0$, $1 \le p \le q \le +\infty$, and $0 \le \lambda \le 1$. Then:*

(i) (Rellich-Kondrachov) If in (5.10) $\ell - \frac{N}{q} < m - \frac{N}{p}$, then the corresponding injection is compact.

(ii) If $m_1 + \nu_1 < m_2 + \nu_2$, then $C^{m_2,\nu_2}(\bar\Omega) \subset C^{m_1,\nu_1}(\bar\Omega)$ with continuous and compact injection.

(iii) If in (5.11) the inequality is strict, then the corresponding injection is compact.

Parts (i) and (iii) hold also if the W-spaces are replaced by the corresponding either W_0- or $W_{\rm loc}$-spaces; in either case Ω may be any domain of $\mathbf{R}^N$.

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(6) By $C^m_b(\Omega)$ we denote the space of functions $\Omega \to \mathbf{C}$ that are continuous and bounded with their derivatives up to order m, possibly without being uniformly continuous. Notice that $C^m_b(\Omega) = C^m(\bar\Omega)$ iff Ω is bounded.)